

MODULI OF FORMAL A -MODULES UNDER CHANGE OF A .

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ABSTRACT. If A is the localization at p or completion at p of a number ring, there exists a Hopf algebroid $(L^A, L^A B)$ classifying one-dimensional formal A -module laws, and a Hopf algebroid $(V^A, V^A T)$ classifying one-dimensional A -typical formal A -module laws. In the paper “Structure of the moduli stack of one-dimensional formal A -modules” we describe some of the structure of these Hopf algebroids; in the present paper we describe what happens to the Hopf algebroids $(L^A, L^A B)$ and $(V^A, V^A T)$ (or equivalently, the moduli stack of one-dimensional formal A -modules), and their cohomology, when we change the number ring A . Most importantly, whenever K/\mathbb{Q}_p is not wildly ramified and A is the ring of integers in K , we provide formulas for the formal A -module Morava stabilizer algebras as quotient Hopf algebras of the classical Morava stabilizer algebras. In future works these formulas will be used to make computations of the cohomology of Morava stabilizer groups of large height.

1. INTRODUCTION.

Let K be a p -adic number ring with ring of integers A . Then there exists a Hopf algebroid $(V^A, V^A T)$ which classifies A -typical one-dimensional formal A -module laws; or equivalently, there exists a moduli stack (in the fpqc topology) \mathcal{M}_{fmA} of one-dimensional formal A -modules. In [11] we worked out some of the structure of $(V^A, V^A T)$ and \mathcal{M}_{fmA} : formulas for the right unit and coproduct maps in $(V^A, V^A T)$, classification of invariant prime ideals in $(V^A, V^A T)$ and an associated A -height stratification of \mathcal{M}_{fmA} , and presentations for the “Morava stabilizer algebras for formal A -modules,” i.e., the Hopf algebras co-representing automorphisms of positive, finite height formal A -modules.

The real motivation for studying $(V^A, V^A T)$ and \mathcal{M}_{fmA} , however, comes from stable homotopy theory: when $A = \mathbb{Z}_{(p)}$, the bigraded cohomology ring

$$\mathrm{Ext}_{\mathrm{graded } V^A T\text{-comodules}}^*(\Sigma^* V^A, V^A) \cong H_{\mathrm{fl}}^*(\mathcal{M}_{fmA}; \omega^*)$$

is the E_2 -term of the Adams-Novikov spectral sequence converging to $\pi_*(S^0)_{(p)}$, the p -local stable homotopy groups of spheres; and it is a simple matter to show that a similar thing is true when $A = \hat{\mathbb{Z}}_p$:

$$\begin{aligned} & \mathrm{Ext}_{\mathrm{graded } V^A T\text{-comodules}}^*(\Sigma^* V^A, V^A) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p \\ & \cong \mathrm{Ext}_{\mathrm{graded } V^{\hat{A}_p} T\text{-comodules}}^*(\Sigma^* V^{\hat{A}_p}, V^{\hat{A}_p}). \end{aligned}$$

A proof is in Prop. 3.2 and a complete statement in terms of the Adams-Novikov E_2 -term is in Cor. 3.3.

We prove that, although the map of Hopf algebroids $(BP_*, BP_* BP) \rightarrow (V^A, V^A T)$ has some bad properties (e.g., the map $BP_* \rightarrow V^A$ is not flat unless $A = \mathbb{Z}_{(p)}$ or $A = \hat{\mathbb{Z}}_p$), it is *split* whenever the fraction field of A is totally ramified over \mathbb{Q}_p .

That is, V^A can be given a BP_*BP -comodule structure such that $(V^A, V^AT) \cong (V^A, V^A \otimes_{BP_*} BP_*BP)$ as graded Hopf algebroids over (BP_*, BP_*BP) , and as a consequence,

$$\mathrm{Ext}_{\mathrm{graded } BP_*BP\text{-comodules}}^*(\Sigma^* BP_*, V^A) \cong \mathrm{Ext}_{\mathrm{graded } V^AT\text{-comodules}}^*(\Sigma^* V^A, V^A).$$

This is proven in Cor. 5.18.

Unfortunately, when A is a nontrivial extension of $\mathbb{Z}_{(p)}$ or $\hat{\mathbb{Z}}_p$, the cohomology ring $\mathrm{Ext}_{\mathrm{graded } V^AT\text{-comodules}}^*(\Sigma^* V^A, V^A)$ *isn't* an Adams-Novikov E_2 -term; see [5] for this negative result.

However, (V^A, V^AT) , for many p -adic number rings A , remains a very useful object for stable homotopy theory. Using the chromatic spectral sequence and several Bockstein spectral sequences (of which there exist analogues for formal A -modules; see [6] or [9]), one reduces the computation of the Adams-Novikov E_2 -term to the computation of the profinite group cohomology of the automorphism groups of formal groups of positive, finite height over $\overline{\mathbb{F}}_p$. Let K be a p -adic number field of degree d over \mathbb{Q}_p , with ring of integers A , and let F be a formal A -module over $\overline{\mathbb{F}}_p$ of A -height h ; then the underlying formal group law of F has p -height dh . We have a classifying map $V^A \rightarrow \overline{\mathbb{F}}_p$, which puts a V^A -algebra structure on $\overline{\mathbb{F}}_p$; and the Hopf algebra

$$\overline{\mathbb{F}}_p \otimes_{V^A} V^AT \otimes_{V^A} \overline{\mathbb{F}}_p,$$

which is the continuous $\overline{\mathbb{F}}_p$ -linear dual of the topological group algebra $\overline{\mathbb{F}}_p[\mathrm{Aut}(F)]$ (where $\mathrm{Aut}(F)$ is the strict automorphism group of F , as a formal A -module), is a quotient Hopf algebra of the Morava stabilizer algebra

$$\overline{\mathbb{F}}_p \otimes_{BP_*} BP_*BP \otimes_{BP_*} \overline{\mathbb{F}}_p$$

which is the continuous $\overline{\mathbb{F}}_p$ -linear dual of the topological group algebra $\overline{\mathbb{F}}_p[\mathrm{Aut}(F')]$ (where $\mathrm{Aut}(F')$ is the strict automorphism group of the underlying formal group of F , forgetting the formal A -module structure). The quotient map

$$(1.1) \quad \overline{\mathbb{F}}_p \otimes_{BP_*} BP_*BP \otimes_{BP_*} \overline{\mathbb{F}}_p \rightarrow \overline{\mathbb{F}}_p \otimes_{V^A} V^AT \otimes_{V^A} \overline{\mathbb{F}}_p$$

is the continuous linear dual of a map of profinite groups (the particular profinite groups involved are described in Prop. 6.1). In [11] we gave a presentation for the Hopf algebra $\overline{\mathbb{F}}_p \otimes_{V^A} V^AT \otimes_{V^A} \overline{\mathbb{F}}_p$, but it was not functorial in the choice of number ring A ; in the present paper, we compute $\overline{\mathbb{F}}_p \otimes_{V^A} V^AT \otimes_{V^A} \overline{\mathbb{F}}_p$ as a quotient of the Morava stabilizer algebra when K/\mathbb{Q}_p is unramified or tamely ramified (Prop. 6.6 and Cor. 6.8). In later works we will make heavy use of a formula for the differentials in a certain May spectral sequence converging to the cohomology of the Morava stabilizer group, at large heights; this formula describes the differentials in terms of classes in the cohomology rings of the quotient Hopf algebras of the Morava stabilizer algebra which are of the form in equation 1.1, i.e., quotient Hopf algebras which are continuous linear dual to automorphism groups of formal A -modules. This is our motivation for the study of the moduli of formal A -modules.

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2. REVIEW OF (V^A, V^AT) .

In this section we review some basic facts about (V^A, V^AT) and Cartier typification. The only new result here is that, if A is the localization at p or the completion at p of a number ring, and M is a graded L^AB -comodule, then

$$\mathrm{Ext}_{\mathrm{graded } L^AB\text{-comodules}}^*(\Sigma^* L^A, M) \cong \mathrm{Ext}_{\mathrm{graded } V^AT\text{-comodules}}^*(\Sigma^* V^A, M \otimes_{L^A} V^A).$$

This generalizes the well-known result that, if M is a graded $(MU_* MU)_{(p)}$ -comodule, then we have an isomorphism

$$\begin{aligned} & \mathrm{Ext}_{\mathrm{graded } (MU_* MU)_{(p)}\text{-comodules}}^{*,*}(\pi_*(MU)_{(p)}, M) \xrightarrow{\cong} \\ & \mathrm{Ext}_{\mathrm{graded } BP_* BP\text{-comodules}}^{*,*}(BP_*, BP_* \otimes_{\pi_*(MU)_{(p)}} M). \end{aligned}$$

Let K be a p -adic number field with ring of integers A . For basic definitions of (one-dimensional) formal A -module laws, A -typicality, and A -height, we refer the reader to [11]. We consider the full Lazard ring

$$L^A \cong A[S_2^A, S_3^A, S_4^A, \dots]$$

and the A -typical Lazard ring

$$V^A \cong A[v_1^A, v_2^A, \dots] \cong A[V_1^A, V_2^A, \dots]$$

with $\{v_i^A\}$ the Araki generators, i.e., if the universal formal A -module law on V^A has fgl-logarithm

$$\lim_{h \rightarrow \infty} p^{-h}[p^h](x) = \log(x) = \sum_{i \geq 0} \ell_i^A x^{p^i},$$

then the log coefficients ℓ_i^A satisfy

$$(2.2) \quad \pi \ell_h^A = \sum_{i=0}^h \ell_i^A (v_{h-i}^A)^{q^i},$$

and $\{V_i^A\}$ the Hazewinkel generators, which satisfy

$$(2.3) \quad \pi \ell_h^A = \sum_{i=0}^{h-1} \ell_i^A (V_{h-i}^A)^{q^i},$$

The Araki v_i^A agrees mod π with the Hazewinkel V_i^A .

Definition 2.1. *These rings have a natural \mathbb{Z} -grading. The gradings are given by*

$$\begin{aligned} |S_i^A| &= 2(i-1) \\ |v_i^A| &= 2(q^i - 1) \\ |V_i^A| &= 2(q^i - 1) \\ |t_i^A| &= 2(q^i - 1) \end{aligned}$$

where q is the cardinality of the residue field of K . These algebras are connected, i.e., the direct summand consisting of elements of degree 0 is precisely the image of the unit map, so we let the augmentation be projection to this direct summand.

The moduli-theoretic interpretation of these gradings is as follows: given a commutative A -algebra R and a ring homomorphism $L^A \xrightarrow{\gamma} R$, let F be the formal A -module law on R classified by γ . Then, if $\alpha \in R^\times$, we get a formal A -module law αF on R given by

$$\alpha F(X, Y) = \alpha^{-1} \log_F^{-1}(\alpha \log_F(X) + \alpha \log_F(Y)).$$

In other words, we have an action of the units R^\times of R on the set of formal A -modules over R . (In the language of stacks, we have an action of the multiplicative group scheme \mathbb{G}_m on the fpqc moduli stack \mathcal{M}_{fmA} of one-dimensional formal A -modules.) The functor taking a commutative A -algebra R to its group of units is co-represented by the Hopf algebra $(A, A[x^{\pm 1}])$, with x grouplike (i.e., $\Delta(x) = x \otimes x$); and we have a map of commutative A -algebras

$$L^A \xrightarrow{\phi} L^A \otimes_A A[x^{\pm 1}]$$

which co-represents the action map of R^\times on the set of one-dimensional formal A -module laws over R . This puts a grading on L^A by letting the summand of L^A in degree i be the subgroup

$$\left\{ a \in L^A : \psi(a) = a \otimes x^{i/2} \right\}.$$

A similar argument holds with V^A in place of L^A .

Remark 2.2. The factor of 2 in the gradings above is due to the graded-commutativity sign convention in algebraic topology and the fact that $V^{\mathbb{Z}_p}$, with the above grading, is isomorphic to the ring of homotopy groups of the p -complete Brown-Peterson spectrum and as such is extremely important to the computation of the stable homotopy groups of spheres.

The fact that the moduli-theoretic interpretation of the \mathbb{Z} -grading on V^A agrees with the one given explicitly above, this essentially follows from the treatment in [4], but we give a more thorough and general treatment of the interrelations between gradings, actions by commutative group schemes, and moduli theory in the paper [10].

Definition 2.3. We consider sequences $I = (i_1, \dots, i_m)$ of positive integers; if $J = (j_1, \dots, j_n)$ is another such sequence then we define $IJ = (i_1, \dots, i_m, j_1, \dots, j_n)$, and this concatenation puts a monoid structure on the set of all such sequences. We define several operators on these sequences:

$$\begin{aligned} |I| &= m, \\ ||I|| &= \sum_{i=1}^m i_m, \end{aligned}$$

and, given a choice of p -adic number ring A , we define the A -valued function Π_A on all such sequences by

$$\begin{aligned} \Pi_A(\emptyset) &= 1, \\ \Pi_A(h) &= \pi_A - \pi_A^{q^h} \text{ for } h \text{ an integer,} \\ \Pi_A(I) &= \Pi_A(||I||) \Pi_A((i_1, i_2, \dots, i_{m-1})). \end{aligned}$$

Proposition 2.4. (1) For each sequence I and number ring A as above and any choice of positive integer m there is a symmetric polynomial $w_I^A = w_I^A(x_1, x_2, \dots, x_m)$, in m variables, of degree $q^{|I|}$, and with coefficients in A , where $w_\emptyset^A = \sum_{t=1}^m x_t$ and

$$(2.4) \quad \sum_{t=1}^m x_t^{q^{|H|}} = \sum_{IJ=H} \frac{\Pi_A(H)}{\Pi_A(I)} (w_J^A)^{q^{|I|}} \text{ for any fixed choice of sequence } H, \text{ and}$$

$$(2.5) \quad w_I^A \equiv (w_{|I|}^A)^{q^{|I|-|I|}} \pmod{\pi_A},$$

where w_i^A (note that the subscript i is an integer, not a sequence of integers!) is the symmetric polynomial defined by

$$\sum_{t=1}^m x_t^{q^k} = \sum_{j=0}^k \pi^j (w_j^A)^{q^{k-j}}$$

with $w_0^A = w_\emptyset^A = \sum_{t=1}^m x_t$. (Note that w_j^A is, in general, not the same as $w_{(j)}^A$.)

Given a sequence (i_1, i_2, \dots, i_m) of positive integers, we define the polynomial $v_I^A \in V^A$ by

$$\begin{aligned} v_\emptyset^A &= 1 \\ v_{(i_1, i_2, \dots, i_m)}^A &= v_{i_1}^A (v_{(i_2, i_3, \dots, i_m)}^A)^{q^{i_1}}. \end{aligned}$$

(2) In terms of Araki generators:

$$\begin{aligned} (\pi - \pi^{q^h}) \ell_h^A &= \sum_{i_1 + \dots + i_r = h} \left(v_{i_1}^A \prod_{j=2}^r \frac{(v_{i_j}^A)^{q^{\sum_{k=1}^{j-1} i_k}}}{\pi - \pi^{q^{\sum_{k=1}^{j-1} i_k}}} \right) \text{ and} \\ \ell_h^A &= \sum_{||I||=h} \frac{v_I^A}{\Pi_A(I)}, \end{aligned}$$

where π is the uniformizer and q the cardinality of the residue field of A , all i_j are positive integers, and the I are sequences of positive integers with v_I^A as above and Π_A as in Def. 2.3.

(3) In terms of Hazewinkel generators:

$$\begin{aligned} \pi \ell_h^A &= \ell_{h-1}^A (V_1^A)^{q^{h-1}} + \dots + \ell_1^A (V^A)_{h-1}^q + V_h^A, \\ \ell_h^A &= \sum_{i_1 + \dots + i_r = h} \pi^{-r} V_{i_1}^A (V_{i_2}^{q^{i_1}}) \dots (V_{i_r}^A)^{q^{i_1 + \dots + i_{r-1}}}, \end{aligned}$$

where π is the uniformizer and q the cardinality of the residue field of A , and all i_j are positive integers.

Proof. Parts one and two of the proposition are proved in [11]. Part three is formula 21.5.4 of [4]. \square

Lemma 2.5. Cartier typification induces a homotopy equivalence. *Let A be the localization at p or the completion at p of a number ring. We have a map*

$$(L^A, L^A B) \rightarrow (V^A, V^A T)$$

of commutative Hopf algebroids, classifying the underlying formal A -module law of the universal A -typical formal A -module law, and this map is a homotopy equivalence of Hopf algebroids, with homotopy inverse given by the Cartier typification map

$$(V^A, V^A T) \rightarrow (L^A, L^A B).$$

Proof. For any commutative A -algebra R , the groupoid $A\text{-typ-fml}A$ of A -typical formal A -module laws over R is a full, faithful subcategory of the groupoid $fmlA$

of formal A -module laws over R :

$$\begin{array}{ccc} A - \text{typ} - fmlA & \xrightarrow{\quad\quad\quad} & fmlA \\ \downarrow \cong & & \downarrow \cong \\ \text{hom}_{\text{Hopf alg'oids}}((V^A, V^AT), (R, R)) & \longrightarrow & \text{hom}_{\text{Hopf alg'oids}}((L^A, L^AB), (R, R)). \end{array}$$

Cartier typification (see 21.7.17 of [4] for Cartier typification of formal A -module laws) gives us a canonical isomorphism of any formal A -module law with a unique A -typical formal A -module law, so $A - \text{typ} - fmlA \hookrightarrow fmlA$ is essentially surjective as well as faithful and full; so it is an equivalence of categories. The functoriality, in R , of Cartier typification means that this equivalence of categories is also an equivalence of Hopf algebroids. \square

Remark 2.6. A map of Hopf algebroids which is a homotopy equivalence, as in the previous proposition, induces, on the associated stacks, a fiberwise equivalence (of groupoids) on sufficiently large fpqc covers. Fiberwise equivalences on sufficiently large fpqc covers are precisely the weak equivalences in any of the “local” model structures on simplicial schemes or cosimplicial commutative rings, i.e., the local model structures on derived stacks (in the fpqc topology) in the sense of [12] and [13].

Corollary 2.7. Cartier typification induces an equivalence of categories of comodules. *Let A be the localization at p or the completion of p of a number ring. Then the functor*

$$M \mapsto M \otimes_{L^A} V^A$$

is an equivalence of categories from the category of L^AB -comodules to the category of V^AT -comodules. As a functor on the category of graded L^AB -comodules, it is an equivalence of categories from the category of graded L^AB -comodules to the category of graded V^AT -comodules.

Corollary 2.8. *Let K be a local number field. Then we have an equivalence of categories between the category of V^AT -comodules and the category of quasicoherent $\mathcal{O}_{\text{fpqc}(\mathcal{M}_{f mA})}$ -modules. This equivalence is cohomology-preserving:*

$$H_{\text{fl}}^*(\mathcal{M}_{f mA}; \tilde{M}) \cong \text{Ext}_{V^AT\text{-comodules}}^{*,*}(V^A, M),$$

where \tilde{M} is the quasicoherent $\mathcal{O}_{\text{fpqc}(\mathcal{M}_{f mA})}$ -module equivalent to the V^AT -comodule M .

Corollary 2.9. Cartier typification induces an iso in cohomology. *Let A be the localization at p or the completion of p of a number ring and let M be a L^AB -comodule. Then we have an isomorphism of graded abelian groups*

$$\begin{aligned} \text{Ext}_{L^AB\text{-comodules}}^*(L^A, M) &\xrightarrow{\cong} \\ \text{Ext}_{V^AT\text{-comodules}}^*(V^A, M \otimes_{L^A} V^A). \end{aligned}$$

If M is a graded L^AB -comodule then we have an isomorphism of bigraded abelian groups

$$\begin{aligned} \text{Ext}_{\text{graded } L^AB\text{-comodules}}^*(\Sigma^* L^A, M) &\xrightarrow{\cong} \\ \text{Ext}_{\text{graded } V^AT\text{-comodules}}^{*,*}(\Sigma^* V^A, M \otimes_{L^A} V^A). \end{aligned}$$

Proof. This is a consequence of Lemma 2.5 together with the observation that, if M is a $L^A B$ -comodule,

$$\begin{aligned} \mathrm{hom}_{L^A B\text{-comodules}}(V^A T \otimes_{V^A} L^A, M) &\cong (V^A T \otimes_{V^A} L^A) \square_{L^A B} M \\ &\cong V^A \otimes_{L^A} M. \end{aligned}$$

See A1.1.6 of [7] for the isomorphism of the cotensor product with hom in the category of comodules over a Hopf algebroid. \square

3. MODULI OF FORMAL A -MODULES UNDER COMPLETION OF A .

In this section we show that, if A is the localization at p of a number ring, and M is a graded $V^A T$ -comodule which is finitely generated as an V^A -module, then

$$\mathrm{Ext}_{\mathrm{graded } V^A T\text{-comodules}}^*(V^A, M) \otimes_A \hat{A}_p \cong \mathrm{Ext}_{\mathrm{graded } V^{\hat{A}_p} T\text{-comodules}}^*(V^{\hat{A}_p}, M \otimes_A \hat{A}_p).$$

Lemma 3.1. *Let R be a \mathbb{Z} -graded commutative ring which is connective, i.e., there exists some integer n such that $R^m \cong 0$ for all $m < n$; furthermore, assume that R^0 is Noetherian and that R^i is a finitely generated R^0 -module for each integer i .*

Then, for any \mathbb{Z} -graded finitely generated R -module M and any ideal I of R generated by elements in R^0 , the natural map

$$\hat{R}_I \otimes_R M \rightarrow \hat{M}_I$$

is an isomorphism of \mathbb{Z} -graded \hat{R}_I -modules.

(This is immediate, when R is Noetherian; the use of this lemma is when R is not Noetherian but R^0 is, e.g. $R \cong BP_$.)*

Proof. Since M is finitely generated as an R -module, M^i is finitely generated as an R^0 -module for any integer i , and since M^i is a finitely generated module over a Noetherian ring, the map $\hat{R}_I^0 \otimes_{R^0} M^i \rightarrow \hat{M}_I^i$ is an isomorphism (see e.g. [2]) for all i . \square

Lemma 3.2. Completion at p commutes with taking coherent cohomology, over a moduli stack of formal modules. *Let A be the localization at p of a number ring. Let N be a finitely generated graded $V^A T$ -comodule; or equivalently, let M be a \mathbb{G}_m -equivariant coherent module over the structure ring sheaf $\mathcal{O}_{\mathrm{fpqc}(\mathcal{M}_{fmA})}$ of the local fpqc site on \mathcal{M}_{fmA} . We have a commutative diagram of isomorphisms of bigraded abelian groups:*

$$\begin{array}{ccc} H_{\mathrm{fl}}^{*,*}(\mathcal{M}_{fmA}; M) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p & & \\ \downarrow \cong & \searrow & \\ H_{\mathrm{fl}}^{*,*}(\mathcal{M}_{fm\hat{A}_p}; M \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p) & & \cong \\ \uparrow \cong & \swarrow & \\ \mathrm{Ext}_{\mathrm{graded } (V^A, V^A T)\text{-comodules}}^{*,*}(\Sigma^* V^A, N) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p & & \\ \downarrow \cong & & \\ \mathrm{Ext}_{\mathrm{graded } (V^{\hat{A}_p}, V^{\hat{A}_p} T)\text{-comodules}}^{*,*}(\Sigma^* V^{\hat{A}_p}, N \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p). & & \end{array}$$

Proof. Due to Lemma 3.1 and our knowledge of the structure of $(V^A, V^A T)$ as a graded Hopf algebroid, we know that

$$\hat{N}_p \cong \hat{A}_p \otimes_A N \quad \widehat{(V^A)}_p \otimes_{V^A} N.$$

The n th comodule in the cobar complex for $(V^A, V^A T)$ with coefficients in N is

$$\left(\bigotimes_{V^A}^n V^A T \right) \otimes_A N,$$

and we have isomorphisms

$$\begin{aligned} & \left(\bigotimes_{V^A}^n V^A T \right) \otimes_{V^A} (N \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p) \\ & \cong \left(\bigotimes_{V^{\hat{A}_p}}^n V^{\hat{A}_p} T \right) \otimes_{V^{\hat{A}_p}} (N \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p), \end{aligned}$$

where the last term is the n th comodule in the cobar complex for $(V^{\hat{A}_p}, V^{\hat{A}_p} T)$ with coefficients in $N \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p$. Now the flatness of $\hat{\mathbb{Z}}_p$ over \mathbb{Z} tells us that tensoring up a cochain complex with $\hat{\mathbb{Z}}_p$ commutes with taking the cohomology of the cochain complex, giving us the desired cohomology isomorphism. \square

Corollary 3.3. Flat cohomology over the moduli stack of formal $\hat{\mathbb{Z}}_p$ -modules computes the p -completion of the Adams-Novikov E_2 -term. *Let X be a topological space or a connective spectrum such that $BP_*(X)$ is finitely generated as a BP_* -module. The E_2 -term of the Adams-Novikov spectral sequence*

$$\mathrm{Ext}_{\mathrm{graded} \, BP_*(BP)\text{-comodules}}^*(\Sigma^* BP_*, BP_*(X)),$$

converging to $\pi_(X)_{(p)}$, admits isomorphisms of bigraded abelian groups*

$$\begin{aligned} & \mathrm{Ext}_{\mathrm{graded} \, BP_*(BP)\text{-comodules}}^*(\Sigma^* BP_*, BP_*(X)) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p \\ & \cong \mathrm{Ext}_{\mathrm{graded} \, V^{\hat{\mathbb{Z}}_p} T\text{-comodules}}^*(\Sigma^* V^{\hat{\mathbb{Z}}_p}, BP_*(X) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p) \\ & \cong H_{\mathrm{fl}}^{*,*}(\mathcal{M}_{f, m\hat{\mathbb{Z}}_p}; M) \end{aligned}$$

where M is the quasicoherent $\mathcal{O}_{\mathrm{fpqc}(\mathcal{M}_{f, m\hat{\mathbb{Z}}_p})}$ -module associated to $BP_(X) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p$.*

4. INJECTIVITY AND SURJECTIVITY OF $(V^A, V^A T) \rightarrow (V^B, V^B T)$.

Suppose we have an extension L/K of p -adic number rings, and suppose that A, B are the rings of integers in K, L , respectively. Then there is an induced map of Lazard rings

$$V^A \otimes_A B \xrightarrow{\gamma} V^B,$$

given by classifying the A -typical formal A -module law underlying the universal B -typical formal B -module law on V^B . In this section we prove that, if L/K is unramified, then γ is surjective, and we compute V^B as a quotient of $V^A \otimes_A B$. We also prove that, if L/K is totally ramified, then γ is injective, and γ is an isomorphism after tensoring with the rationals.

Proposition 4.1. *Let K, L be p -adic number fields, and let L/K be unramified of degree f , both fields having uniformizer π , and let q be the cardinality of the residue field of K . Using the Hazewinkel generators for V^A and V^B , the map $V^A \otimes_A B \xrightarrow{\gamma} V^B$ is determined by*

$$\gamma(V_i^A) = \begin{cases} V_{i/f}^B & \text{if } f \mid i \\ 0 & \text{if } f \nmid i \end{cases}.$$

Proof. First, since $\gamma(\ell_i^A) = 0$ if $f \nmid i$, we have that $\gamma(V_i^A) = 0$ for $i < f$, and

$$\begin{aligned} \frac{1}{\pi} V_1^B &= \ell_1^B \\ &= \gamma(\ell_f^A) \\ &= \gamma\left(\frac{1}{\pi} \sum_{i=0}^{f-1} \ell_i^A (V_{f-i}^A)^{q^i}\right) \\ &= \frac{1}{\pi} \gamma(V_f^A), \end{aligned}$$

so $\gamma(V_f^A) = V_1^B$.

We proceed by induction. Suppose that $\gamma(V_i^A) = 0$ if $f \nmid i$ and $i < hf$ and $\gamma(V_i^A) = V_{i/f}^B$ if $f \mid i$ and $i < hf$. Then

$$\begin{aligned} \gamma(\ell_{hf}^A) &= \gamma\left(\sum_{i=0}^{hf-1} \ell_i^A (V_{hf-i}^A)^{q^i}\right) \\ &= \sum_{i=0}^{h-1} \ell_i^B \gamma(V_{(h-i)f}^A)^{q^{if}} \\ &= \gamma(V_{hf}^A) + \sum_{i=1}^{h-1} \ell_i^B (V_{h-i}^B)^{q^{if}} \\ &= \ell_h^B = V_h^B + \sum_{i=1}^{h-1} \ell_i^B (V_{h-i}^B)^{q^{if}}, \end{aligned}$$

so $\gamma(V_{hf}^A) = V_h^B$.

Now suppose that $0 < j < f$ and $\gamma(V_i^A) = 0$ if $f \nmid i$ and $i < hf + j$ and $\gamma(V_i^A) = V_{i/f}^B$ if $f \mid i$ and $i < hf$. We want to show that $\gamma(V_{hf+j}^A) = 0$. We see that

$$\begin{aligned} 0 &= \gamma(\ell_{hf+j}^A) \\ &= \gamma\left(\sum_{i=0}^{hf+j-1} \ell_i^A (V_{hf+j-i}^A)^{q^i}\right) \\ &= \sum_{i=0}^h \ell_i^B \gamma(V_{(h-i)f+j}^A)^{q^{if}} \\ &= \gamma(V_{hf+j}^A). \end{aligned}$$

This concludes the induction. □

Corollary 4.2. *Let L/K be unramified of degree f . Then*

$$V^B \cong (V^A \otimes_A B) / (\{V_i^A : f \nmid i\}).$$

When L/K is ramified, the map γ is much more complicated. In general, when $K(B)/K(A)$ is totally ramified, one can always use the above formulas to solve the equation

$$\gamma(\ell_i^A) = \ell_i^B$$

to get a formula for $\gamma(v_i^A)$ or $\gamma(V_i^A)$, as long as one knows $\gamma(v_j^A)$ or $\gamma(V_j^A)$ for all $j < i$; but we do not know of a simple closed form for $\gamma(v_i^A)$ for general i . We give formulas for low i :

Proposition 4.3. *Let K, L be p -adic number fields with rings of integers A and B , and let L/K be totally ramified, both fields having residue field \mathbb{F}_q . Let π_A, π_B be uniformizers for A, B , respectively. Let γ be the ring homomorphism $V^A \otimes_A B \xrightarrow{\gamma} V^B$ classifying the underlying formal A -module law of the universal formal B -module law. Then, in terms of Araki generators,*

$$\begin{aligned} \gamma(v_1^A) &= \frac{\pi_A - \pi_A^q}{\pi_B - \pi_B^q} v_1^B, \\ \gamma(v_2^A) &= \frac{\pi_A - \pi_A^{q^2}}{\pi_B - \pi_B^{q^2}} v_2^B \\ &= + \left(\frac{1}{\pi_B - \pi_B^q} \right) \left(\frac{\pi_A - \pi_A^{q^2}}{\pi_B - \pi_B^{q^2}} - 1 \right) (v_1^B)^{q+1}, \end{aligned}$$

and in terms of Hazewinkel generators,

$$\begin{aligned} \gamma(V_1^A) &= \frac{\pi_A}{\pi_B} V_1^B, \\ \gamma(V_2^A) &= \frac{\pi_A}{\pi_B} V_2^B + \left(\frac{\pi_A}{\pi_B^2} - \frac{\pi_A^q}{\pi_B^{q+1}} \right) (V_1^B)^{q+1}. \end{aligned}$$

Proof. These formulas follow immediately from setting $\gamma(\ell_i^A) = \ell_i^B$ and solving, using the formulas in Prop. 2.4. \square

Proposition 4.4. *Let K, L be p -adic number fields with rings of integers A, B , respectively, and let L/K be totally ramified. Write γ for the ring map*

$$V^A \otimes_A B \rightarrow V^B.$$

Then

$$V^A \otimes_A B \otimes_B L \xrightarrow{\gamma \otimes_B L} V^B \otimes_B L$$

is surjective.

Proof. We will use Hazewinkel generators here. Given any V_i^B , we want to produce some element of $V^A \otimes_A B$ which maps to it. We begin with $i = 1$. Since $\ell_1^A = \pi_K^{-1} V_1^A$ and $\ell_1^B = \pi_L^{-1} V_1^B$ we have $\gamma(\frac{\pi_L}{\pi_K} V_1^A) = V_1^B$.

Now we proceed by induction. Suppose that we have shown that there is an element in $(V_1^A, \dots, V_{j-1}^A) \subseteq V^A \otimes_A B$ which maps via γ to V_{j-1}^B . Then

$$\gamma(\pi_K^{-1} \sum_{i=0}^{j-1} \ell_i^A (V_{j-i}^A)^{q^i}) = \pi_L^{-1} \sum_{i=0}^{j-1} \ell_i^B (V_{j-i}^B)^{q^i}$$

and hence

$$\begin{aligned}
 \gamma(\pi_K^{-1}V_j^A) &= \pi_L^{-1} \sum_{i=0}^{j-1} \ell_i^B (V_{j-i}^B)^{q^i} - \gamma(\pi_K^{-1} \sum_{i=1}^{j-1} \ell_i^A (V_{j-i}^A)^{q^i}) \\
 (4.6) \qquad &\equiv \pi_L^{-1} V_j^B \pmod{(V_1^B, V_2^B, \dots, V_{j-1}^B)}.
 \end{aligned}$$

So $\gamma(\frac{\pi_L}{\pi_K} V_j^A) \equiv V_j^B$ modulo terms hit by elements in the ideal generated by Hazewinkel generators of lower degree. This completes the induction. \square

Corollary 4.5. *Let K, L be p -adic number fields with rings of integers A, B and uniformizers π_K, π_L , respectively, and let L/K be a totally ramified, finite extension. Choose an $x \in V^B$. Then there exists some integer a such that*

$$\pi_L^a x \in \text{im}(V^A \otimes_A B \xrightarrow{\gamma} V^B).$$

For the proof of the next proposition we will use a monomial ordering on V^A .

Definition 4.6. *We put the following ordering on the Hazewinkel generators of V^A :*

$$V_i^A \leq V_j^A \quad \text{iff} \quad i \leq j$$

and we put the lexicographic order on the monomials of V^A .

Since this total ordering on the generators of V^A, V^B is preserved by $V^A \otimes_A B \xrightarrow{\gamma} V^B$ when L/K is totally ramified (equation 4.6), the ordering on the monomials of V^A, V^B is also preserved by γ . The ordering on the monomials in V^A is a total ordering (see e.g. [3]).

Proposition 4.7. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be a totally ramified, finite extension. Then $V^A \otimes_A B \xrightarrow{\gamma} V^B$ is injective.*

Proof. Suppose $x \in \ker \gamma$. Let x_0 be the sum of the monomial terms of x of highest order in the lexicographic ordering. Then, since γ preserves the ordering of monomials, $\gamma(x_0) = 0$. However, since the lexicographic ordering on V^A is a total ordering, x_0 is a monomial; let $x_0 = \prod_{i \in I} (V_i^A)^{\epsilon_i}$ for some set I of positive integers and positive integers $\{\epsilon_i\}_{i \in I}$. Then the terms of $\gamma(x_0)$ of highest order in the lexicographic ordering on V^B consist of just the monomial $\prod_{i \in I} (V_i^B)^{\epsilon_i}$ (equation 4.6). Since $\gamma(x_0) = 0$ this implies that $\prod_{i \in I} (V_i^B)^{\epsilon_i} = 0$, i.e., $x_0 = 0$ and finally $x = 0$. \square

5. SPLITTINGS OF HOPF ALGEBROIDS CLASSIFYING FORMAL A-MODULES.

In this section, we prove that, when A, B are the rings of integers in p -adic number fields K, L , and L/K is a field extension, then the Hopf algebroid $(L^B, L^B B)$ classifying formal B -module laws splits as a Hopf algebroid:

$$(L^B, L^B B) \cong (L^B, L^B \otimes_{L^A} L^A B),$$

and when L/K is totally ramified, then the Hopf algebroid classifying B -typical formal B -module laws also splits as a Hopf algebroid:

$$(V^B, V^B T) \cong (V^B, V^B \otimes_{V^A} V^A T).$$

This implies that we have isomorphisms in cohomology:

$$\text{Cotor}_{L^B B}^{*,*}(L^B, L^B) \cong \text{Cotor}_{L^A B}^{*,*}(L^A, L^B),$$

and when L/K is totally ramified,

$$\mathrm{Cotor}_{V^B T}^{*,*}(V^B, V^B) \cong \mathrm{Cotor}_{V^A T}^{*,*}(V^A, V^B).$$

We arrive at these facts in Cor. 5.18. When L/K is unramified and of degree > 1 , the morphism of Hopf algebroids

$$(V^A, V^A T) \rightarrow (V^B, V^B T)$$

is not split, but still induces an isomorphism in cohomology

$$\mathrm{Cotor}_{V^A T}^{*,*}(\Sigma^* V^A, (V^A T \otimes_{V^A} V^B) \square_{V^B T} V^B) \cong \mathrm{Cotor}_{V^B T}^{*,*}(\Sigma^* V^B, V^B).$$

This is proven in Cor. 5.22.

Proposition 5.1. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be a totally ramified, finite extension, and let F be an A -typical formal A -module over a commutative B -algebra R . If F admits an extension to a B -typical formal B -module (i.e., a factorization of the structure map $A \xrightarrow{\rho} \mathrm{End}(F)$ through B) then that extension is unique.*

Proof. We need to show that, given maps

$$\begin{array}{ccc} V^A \otimes_A B & \xrightarrow{\gamma} & V^B \\ & \searrow \theta & \\ & & R \end{array}$$

where θ is the classifying map of F , if there exists a map $V^B \xrightarrow{g} R$ making the diagram commute, then that map g is unique.

The map g is determined by its values on the generators V_i^B of V^B . For any choice of i , let a be an integer such that $\pi_L^a V_i^B \in \mathrm{im} \gamma$ (the existence of such an a is guaranteed by Cor. 4.5). Now let $\pi_L^a \bar{V}_i^B$ be a lift of $\pi_L^a V_i^B$ to $V^A \otimes_A B$, and in order for the diagram to commute, $g(\pi_L^a \bar{V}_i^B)$ must be equal to $f(\pi_L^a \bar{V}_i^B)$, so

$$g(V_i^B) = \pi_L^{-a} f(\pi_L^a \bar{V}_i^B),$$

completely determining g . Hence g is unique. \square

Corollary 5.2. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be a finite extension. Let F be an A -typical formal A -module law over a commutative B -algebra R . Then, if F admits an extension to an B -typical formal B -module law, that extension is unique.*

Proof. Let L_{nr} be the maximal subextension of L which is unramified over K , and let B_{nr} be its ring of integers. Then $V^A \rightarrow V^{B_{\mathrm{nr}}}$ is surjective, so any map $V^A \otimes_A B \rightarrow R$ admitting a factorization through $V^A \otimes_A B \rightarrow V^{B_{\mathrm{nr}}} \otimes_{B_{\mathrm{nr}}} B$ admits only one such factorization, i.e., if there is an B_{nr} -typical formal B_{nr} -module law extending F , it is unique.

Now we use Cor. 5.1 to see that if there is an extension of this B_{nr} -typical formal B_{nr} -module law to a B -typical formal B -module law, then that B -typical formal B -module law is unique. \square

We will see that, when L/K is unramified, there may exist multiple extensions of the structure map $A \xrightarrow{\rho} \mathrm{End}(F)$ of a A -typical formal A -module law to the structure map $B \rightarrow \mathrm{End}(F)$ of a formal B -module law, but as a result of the

previous proposition, only one of these extensions yields an B -typical formal B -module law.

Proposition 5.3. *Let K be a p -adic number field with ring of integers A , let F be a p -typical formal A -module law with a logarithm, and let p^f be the cardinality of the residue field of A . Recall that, if $\alpha \in A$, then $[\alpha]_F(X)$ is the usual notation for the endomorphism of F which is the image of α under the formal A -module structure map $A \xrightarrow{\rho} \text{End}(F)$. Then the following conditions are equivalent:*

- (1) $[\zeta]_F(X) = \zeta X$ for some primitive $(p^f - 1)$ th root of unity $\zeta \in A$.
- (2) F is A -typical.
- (3) $[\zeta]_F(X) = \zeta X$ for all $(p^f - 1)$ th roots of unity $\zeta \in A$.
- (4) The following diagram is commutative:

$$\begin{array}{ccc} \mu_{(p)}(A) & \xrightarrow{\rho|_{\mu_{(p)}(A)}} & \text{Aut}(F) , \\ & \searrow s & \downarrow \\ & & PS(A) \end{array}$$

where $\mu_{(p)}(A)$ is the group of roots of unity in A of order prime to p , and $PS(A)$ is the group (under composition) of power series in a single variable with coefficients in A .

Proof. **Condition 1 implies condition 2:** Let R be the ring over which F is defined and let $\{u_i\}$ be the log coefficients of F . Since $[\zeta]_F$ is by definition $\log_F^{-1}(\zeta \log_F(X))$, assuming $\zeta X = [\zeta]_F(X)$ gives us

$$\begin{aligned} \sum_{i \geq 0} \zeta^{p^i} u_i X^{p^i} &= \sum_{i \geq 0} u_i (\zeta X)^{p^i} \\ &= \log_F(\zeta X) \\ &= \log_F([\zeta]_F(X)) \\ &= \zeta \log_F(X) \\ &= \sum_{i \geq 0} \zeta u_i X^i, \end{aligned}$$

so $u_i = 0$ for all i such that $\zeta^{p^i} \neq \zeta$, i.e., all i such that $(p^f - 1) \nmid (p^i - 1)$. We have factorizations $p^f - 1 = \prod_{d|f} \Phi_d(p)$ and $p^i - 1 = \prod_{d|i} \Phi_d(p)$, where $\Phi_d(X)$ is the d th cyclotomic polynomial, which is irreducible. These factorizations are unique factorizations since $\mathbb{Z}[X]$ is a UFD, so when $f \nmid i$, we see that $\Phi_{\frac{f}{\gcd(f,i)}}(p)$ appears in the factorization of $p^f - 1$ but not in the factorization of $p^i - 1$. Hence $(p^f - 1) | (p^i - 1)$ only if $f | i$, and $u_i = 0$ if $f \nmid i$. This tells us that $\log_F(X) = \sum_{i \geq 0} u_{fi} X^{p^{fi}}$, so F is A -typical.

Condition 2 implies condition 3: We know that F 's logarithm is of the form $\log_F(X) = \sum_{i \geq 0} u_i X^{p^{fi}}$ for some $\{u_i\}$. Choose a $p^f - 1$ th root of

unity $\zeta \in o_K$. Now

$$\begin{aligned}
[\zeta]_F(X) &= \log_F^{-1}(\zeta \log_F(X)) \\
&= \log_F^{-1}(\zeta \sum_{i \geq 0} u_i X^{p^{f_i}}) \\
&= \log_F^{-1}(\sum_{i \geq 0} u_i \zeta^{p^{f_i}} X^{p^{f_i}}) \\
&= \log_F^{-1}(\log_F(\zeta X)) = \zeta X.
\end{aligned}$$

Condition 3 implies condition 1: This is immediate.

Condition 3 is equivalent to condition 4: Every element in $\mu_{(p)}(A)$ is a $(p^f - 1)$ th root of unity (see 2.4.3 Prop. 2 of [8]). Let $\zeta \in \mu_{(p)}(A)$, and now $s(\zeta) = \zeta(X)$ while ζ 's image under the composite map

$$\mu_{(p)}(A) \hookrightarrow \text{Aut}(F) \hookrightarrow PS(A)$$

is the power series $[\zeta]_F(X)$.

□

Proposition 5.4. *Let K, L be p -adic number fields with rings of integers A, B , let L/K be a totally ramified extension, and let R be a commutative B -algebra. If F is an A -typical formal B -module law over R with a logarithm, then it is B -typical.*

Proof. Since L/K is totally ramified, $\log_F(X) = \sum_{i \geq 0} l_i X^{q^{f_i}}$, where q^f is the cardinality of the residue fields of both A and B ; so it is immediate that F is B -typical if it is A -typical. □

Now we remove the hypothesis that the formal module has a logarithm.

Proposition 5.5. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be a totally ramified extension and let R be a commutative B -algebra. If F is an A -typical formal B -module law over R , then it is B -typical.*

Proof. Let S be an B -algebra which surjects on to R and such that $S \rightarrow S \otimes_B L$ is injective. Such an algebra always exists, e.g. $B[Z_a : a \in R]$ (this strategy of proof is adapted from 21.7.18 of [4]). Let $S \xrightarrow{w} R$ be a surjection. We choose a lift \bar{F} of F to S , and since L^B and $V^A \otimes_A B$ are free B -algebras, we can choose lifts $L^B \rightarrow S$, $V^A \otimes_A B \rightarrow S$ of the classifying maps of F as a formal B -module law and as an A -typical formal A -module law, respectively. Now by construction \bar{F} is a A -typical formal B -module law with a logarithm, so it is also B -typical and there exists a map $V^B \rightarrow S$ classifying it; composing this map with w gives us F as an B -typical formal B -module law. We include a diagram showing all these maps:

$$\begin{array}{ccc}
L^A \otimes_A B & \longrightarrow & V^A \otimes_A B \\
\downarrow & & \downarrow \\
L^B & \longrightarrow & V^B \\
& \searrow & \searrow \\
& & S \xrightarrow{w} R
\end{array}$$

□

Corollary 5.6. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be a totally ramified extension. Then $V^B \cong L^B \otimes_{L^A} V^A$.*

Proof. Given a commutative B -algebra A with an A -typical formal B -module law F on it, i.e., maps making the following diagram commute:

$$\begin{array}{ccc} L^A \otimes_A B & \longrightarrow & V^A \otimes_A B \\ \downarrow & & \downarrow \\ L^B & \longrightarrow & V^B \\ & \searrow & \downarrow \\ & & A \end{array},$$

F is B -typical, i.e., there exists a unique map $V^B \rightarrow A$ making the above diagram commute. So the square in the above diagram is a pushout square in commutative B -algebras, i.e.,

$$\begin{aligned} V^B &\cong L^B \otimes_{L^A \otimes_A B} (V^A \otimes_A B) \\ &\cong L^B \otimes_{L^A} V^A. \end{aligned}$$

□

The situation is very different for L/K unramified.

Proposition 5.7. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be unramified of degree f . Then $V^A \otimes_{L^A} L^B$ surjects on to V^B , but this map is not an isomorphism unless $f = 1$.*

Proof. We consider the following diagrams in the category of graded commutative B -algebras:

$$(5.7) \quad \begin{array}{ccc} & & V^A \otimes_A B \\ & \nearrow \Theta & \\ L^A \otimes_A B & & \\ & \searrow & \\ & & L^B \end{array}$$

$$(5.8) \quad \begin{array}{ccc} & & B \\ & \nearrow & \\ B & & \\ & \searrow & \\ & & V^B \end{array}$$

We refer to diagram (5.7) as X_1 and diagram (5.8) as X_2 . There is a map $X_1 \rightarrow X_2$ given by Θ and the augmentations and a map $X_2 \rightarrow X_1$ given by Cartier typicalization and the unit maps, and it is trivial to check that the composite $X_2 \rightarrow X_1 \rightarrow X_2$ is the identity on X_2 and that the map $\text{colim } X_1 \rightarrow \text{colim } X_2$ is the map produced via the universal property of the pushout of X_1 by mapping

L^B and $V^A \otimes_A B$ to V^B . Hence $\text{colim } X_2 \cong V^B$ is a retract of $\text{colim } X_1 \cong (V^A \otimes_A B) \otimes_{L^A \otimes_A B} L^B$, and the map $(V^A \otimes_A B) \otimes_{L^A \otimes_A B} L^B \xrightarrow{g} V^B$ is a surjection.

However, when $f > 1$ the map g is not an isomorphism: we let γ be the map $V^A \otimes_A B \xrightarrow{\gamma} V^B$, and then it is certainly true that $\gamma(V_1^A) = 0$, by Prop. 4.1, so $g(1 \otimes V_1^A) = 0$. However, we show that $1 \otimes V_1^A \neq 0 \in (V^A \otimes_A B) \otimes_{L^A \otimes_A B} L^B$. Considering $(V^A \otimes_A B) \otimes_{L^A \otimes_A B} L^B$ as a quotient of $(V^A \otimes_A B) \otimes_B L^B \cong B[S_2^B, S_3^B, \dots][V_1^A, V_2^A, \dots]$, all the $S_i^B \otimes 1$ are identified with zero for $q \nmid i$, but $\pi S_q^B \otimes 1 \sim 1 \otimes V_1^A$ since S_q^A maps to both in the appropriate pushout diagram, and since all involved maps are graded, there are no other elements in $L^A \otimes_A B$ which can map to $V_1^A \in V^A \otimes_A B$. So $1 \otimes V_1^A$ is nonzero in $L^B \otimes_{L^A} V^A$, although it is divisible by π there. \square

Corollary 5.8. *When L/K is unramified and nontrivial there exists at least one A -typical formal B -module which is not B -typical. In particular, the universal A -typical formal B -module law on $L^B \otimes_{L^A} V^A$ is not B -typical.*

Definition 5.9. (Extending a formal module structure via an isomorphism.) *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be an extension. Given an isomorphism of formal A -modules $F \xrightarrow{\phi} G$ and a formal B -module structure on F , there is a natural formal B -module structure on G given by*

$$\begin{aligned} B &\xrightarrow{\rho_G} \text{End}(G) \\ \alpha &\mapsto \phi \circ \rho_F(\alpha) \circ \phi^{-1}. \end{aligned}$$

Since the data of a formal B -module law F over R , a formal A -module law G over R , and a strict isomorphism $F \rightarrow G$ of formal A -modules over R is equivalent to a map from the diagram

$$\begin{array}{ccc} L^A & \xrightarrow{\gamma_\sigma} & L^B \\ \downarrow \eta_L & & \\ L^A B & & \\ \uparrow \eta_R & & \\ L^A & & \end{array}$$

to R , the above definition gives us a ring map from L^B to $L^B \otimes_{L^A} L^A B$ classifying the formal B -module law obtained by the definition. We will call this map $L^B \xrightarrow{\psi} L^B \otimes_{L^A} L^A B$. When F is B -typical and G is A -typical, we have a priori only a ring map $L^B \rightarrow V^B \otimes_{V^A} V^A T$, but when L/K is totally ramified then we know that the extension of G to a formal B -module law is B -typical (Prop. 5.4), so when L/K is totally ramified we have a ring map $V^B \xrightarrow{\psi'} V^B \otimes_{V^A} V^A T$.

Proposition 5.10. *Let (R, Γ) be a commutative bialgebroid over a commutative ring A , and let S be a right Γ -comodule algebra, such that the following diagram commutes:*

$$\begin{array}{ccc} R & \xrightarrow{\eta_R} & \Gamma \\ \downarrow f & & \downarrow f \otimes \text{rid}_\Gamma \\ S & \xrightarrow{\psi} & S \otimes_R \Gamma \end{array}$$

where f is the R -algebra structure map $R \xrightarrow{f} S$. Then the algebraic object given by the pair $(S, S \otimes_R \Gamma)$, with its right unit $S \rightarrow S \otimes_R \Gamma$ equal to the comodule structure map on S , is a bialgebroid over A . If (R, Γ) is a Hopf algebroid, then so is $(S, S \otimes_R \Gamma)$; if (R, Γ) is a graded bialgebroid and S is a graded right Γ -comodule, then so is $(S, S \otimes_R \Gamma)$; if (R, Γ) is a graded Hopf algebroid and S is a graded right Γ -comodule, then so is $(S, S \otimes_R \Gamma)$.

Proof. Note that the condition on the comodule algebra S guarantees that ψ “extends” η_R in a way that allows us to define a coproduct on $(S, S \otimes_R \Gamma)$, as we need an isomorphism $S \otimes_R \Gamma \otimes_R \Gamma \cong (S \otimes_R \Gamma) \otimes_S (S \otimes_R \Gamma)$.

First, we make explicit the structure maps on $(S, S \otimes_R \Gamma)$ (we note that, throughout this proof, we will consistently use the symbols η_L, η_R, ϵ , and Δ (and χ if (R, Γ) is a Hopf algebroid) to denote the structure maps on (R, Γ) , and $\psi : S \rightarrow S \otimes_R \Gamma$ to denote the comodule structure map of S):

$$\begin{aligned}
 \text{augmentation:} \quad & S \otimes_R \Gamma \xrightarrow{\text{id}_S \otimes_R \epsilon} S \\
 \text{left unit:} \quad & S \xrightarrow{\text{id}_S \otimes_R \eta_L} S \otimes_R \Gamma \\
 \text{right unit:} \quad & S \xrightarrow{\psi} S \otimes_R \Gamma \\
 \text{coproduct:} \quad & S \otimes_R \Gamma \xrightarrow{\text{id}_S \otimes_R \Delta} S \otimes_R \Gamma \otimes_R \Gamma \xrightarrow{\cong} (S \otimes_R \Gamma) \otimes_S (S \otimes_R \Gamma) \\
 \text{conjugation,} \\
 \text{when } (R, \Gamma) \text{ is a} \quad & S \otimes_R \Gamma \xrightarrow{\text{id}_S \otimes_R \chi} S \otimes_R \Gamma. \\
 \text{Hopf algebroid:}
 \end{aligned}$$

We now show that these structure maps satisfy the axioms for being a bialgebroid. First we show that the coproduct on $(S, S \otimes_R \Gamma)$ is a left S -module morphism, i.e., that this diagram commutes:

$$\begin{array}{ccc}
 S & \xrightarrow{\cong} & S \otimes_R R \otimes_R R \xrightarrow{\text{id}_S \otimes_R \eta_L \otimes_R \eta_L} S \otimes_R \Gamma \otimes_R \Gamma \\
 \downarrow \text{id}_S \otimes_R \eta_L & & \downarrow \cong \\
 S \otimes_R \Gamma & \xrightarrow{\text{id}_S \otimes_R \Delta} & S \otimes_R \Gamma \otimes_R \Gamma \xrightarrow{\cong} (S \otimes_R \Gamma) \otimes_S (S \otimes_R \Gamma),
 \end{array}$$

whose commutativity follows from Δ being a left R -module morphism.

We now check that the coproduct on $(S, S \otimes_R \Gamma)$ is also a right S -module morphism:

$$\begin{array}{ccc}
 S & \xrightarrow{\cong} & S \otimes_S S \\
 \downarrow \psi & & \downarrow \psi \otimes_S \psi \\
 S \otimes_R \Gamma & \xrightarrow{\text{id}_S \otimes_R \Delta} & S \otimes_R \Gamma \otimes_R \Gamma \xrightarrow{\cong} (S \otimes_R \Gamma) \otimes_S (S \otimes_R \Gamma),
 \end{array}$$

whose commutativity follows from $(\text{id}_S \otimes_R \Delta) \circ \psi = (\psi \otimes_R \text{id}_\Gamma) \otimes_R \psi$, one of the axioms for S being a Γ -comodule.

We now check that the augmentation on $(S, S \otimes_R \Gamma)$ is a left S -module morphism, i.e., $\text{id}_S = (\text{id}_S \otimes_R \epsilon) \circ (\text{id}_S \otimes_R \eta_L)$, which follows immediately from $\text{id}_R = \epsilon \circ \eta_L$; and we check that the augmentation on $(S, S \otimes_R \Gamma)$ is a right S -module morphism, i.e., $(\text{id}_S \otimes_R \epsilon) \circ \psi = \text{id}_S$, which is precisely the other axiom for S being a Γ -comodule.

That the diagram

$$\begin{array}{ccc} S & \xrightarrow{\text{id}_S \otimes_R \Delta} & S \otimes_R \Gamma \otimes_R \Gamma \\ \downarrow \text{id}_S \otimes_R \Delta & & \downarrow \text{id}_S \otimes_R \text{id}_\Gamma \otimes_R \epsilon \\ S \otimes_R \Gamma \otimes_R \Gamma & \xrightarrow{\text{id}_S \otimes_R \epsilon \otimes_R \text{id}_\Gamma} & S \otimes_R \Gamma, \end{array}$$

commutes follows from the analogous property being satisfied by (R, Γ) .

The last property we need to verify is the commutativity of the diagram:

$$\begin{array}{ccc} S \otimes_R \Gamma & \xrightarrow{\text{id}_S \otimes_R \Delta} & S \otimes_R \Gamma \otimes_R \Gamma \\ \downarrow \text{id}_S \otimes_R \Delta & & \downarrow \text{id}_S \otimes_R \Delta \otimes_R \text{id}_\Gamma \\ S \otimes_R \Gamma \otimes_R \Gamma & \xrightarrow{\text{id}_S \otimes_R \text{id}_\Gamma \otimes_R \Delta} & S \otimes_R \Gamma \otimes_R \Gamma \otimes_R \Gamma, \end{array}$$

which again follows immediately from the analogous property for (R, Γ) .

In the graded cases, it is very easy to check by inspection of the above structure maps and diagrams that, since ψ is graded and all structure maps of (R, Γ) are graded maps, $(S, S \otimes_R \Gamma)$ and its structure maps are graded.

This proof has been put in terms of a right Γ -comodule algebra and $(S, S \otimes_R \Gamma)$ but the same methods work with obvious minor changes to give the stated result in terms of a left Γ -comodule algebra and $(S, \Gamma \otimes_R S)$. \square

Remark 5.11. Let A be a commutative ring, let (R, Γ) be a commutative Hopf algebroid over A , and let S be a commutative R -algebra. Write x for the stack associated to (R, Γ) ; then a right Γ -comodule algebra structure map on S satisfying condition 5.10 is precisely a descent datum for $\text{Spec } S$ for the cover $\text{Spec } R \rightarrow x$; i.e., S together with a right Γ -comodule algebra structure map on S satisfying condition 5.10 uniquely (up to homotopy) determines a stack \mathcal{Y} over x such that the canonical map from $\text{Spec } S$ to the homotopy fiber product $\text{Spec } R \times_x \mathcal{Y}$ is an isomorphism.

Proposition 5.12. (1) *The map ψ from Def. 5.9 is a right $L^A T$ -comodule algebra structure map on L^B , and it satisfies condition 5.10.*
 (2) *The map ψ' from Def. 5.9 (only defined when L/K is totally ramified) is a right $V^A T$ -comodule structure map on V^B , and it satisfies condition 5.10.*

Proof. If (R, Γ) is a Hopf algebroid over A and M is an R -module, then an A -module map $M \xrightarrow{\phi} M \otimes_A \Gamma$ is a right Γ -comodule structure map on M if and only if ϕ satisfies a counitality condition and a coassociativity condition; see Appendix A of [7] for these conditions. In the case of the particular maps ψ and ψ' in the statement of the proposition, the counitality condition is equivalent, to the extension of a formal B -module law F over the isomorphism id_R being again F itself, which is clearly true; and the coassociativity condition is equivalent to the following: given a formal B -module law F_1 and formal A -module laws F_2, F_3 , and

strict isomorphisms of formal A -module laws $F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} F_3$, extending F_2 to a formal B -module law over f_1 and then extending F_3 to a formal B -module law over f_2 gives us the same formal B -module law as extending F_3 to a formal B -module law over $f_2 \circ f_1$. In the first case, the relevant structure map is

$$\begin{aligned} B &\xrightarrow{\rho} \text{End}(F_3) \\ \alpha &\mapsto f_2 \circ (f_1 \circ \rho_{F_1}(\alpha) \circ f_1^{-1}) \circ f_2^{-1}, \end{aligned}$$

and in the second case, the relevant structure map is

$$\begin{aligned} B &\xrightarrow{\rho'} \text{End } F_3 \\ \alpha &\mapsto (f_2 \circ f_1) \circ \rho_{F_1}(\alpha) \circ (f_2 \circ f_1)^{-1}. \end{aligned}$$

Now $\rho = \rho'$, giving us the coassociativity condition.

For the first case, the comodule algebra L^B , we note that condition 5.10 is equivalent to the following statement: given a formal B -module law F , a formal A -module law G , and a strict isomorphism of formal A -module laws $F \xrightarrow{f} G$, if we extend G to a formal B -module law via f , its underlying formal A -module law is G itself, and this is clearly true. For the second case, the comodule algebra V^B , we note that condition 5.10 is equivalent to the following statement: given a B -typical formal B -module law F , an A -typical formal A -module law G , and a strict isomorphism of A -typical formal A -module laws $F \xrightarrow{f} G$, if we extend G to a B -typical formal B -module law via f , its underlying A -typical formal A -module law is G itself, and this is again clearly true. \square

Proposition 5.13. (Tensoring up, in cohomology.) *Let A be a commutative ring and let $(R, \Gamma) \xrightarrow{f} (S, \Sigma)$ be a split map of graded Hopf algebroids over A . (A map of Hopf algebroids as above is said to be split if S is a Γ -comodule algebra and $\Sigma \cong \Gamma \otimes_R S$ or, equivalently, $\Sigma \cong S \otimes_R \Gamma$). Let N be a left Σ -comodule. Then there is an isomorphism*

$$\text{Cotor}_{\Gamma}^{*,*}(R, N) \cong \text{Cotor}_{\Sigma}^{*,*}(S, N).$$

Proof. The $E_2^{*,*,*}$ of the change-of-rings spectral sequence (see A.3.11 of [7]) for f is

$$\text{Cotor}_{\Gamma}^{r,t}(R, \text{Cotor}_{\Sigma}^{s,t}(\Gamma \otimes_R S, N)) \cong \text{Cotor}_{\Gamma}^{r,t}(R, N),$$

since $\Gamma \otimes_R S \cong \Sigma$ is free, hence relatively injective, as a left Σ -comodule. This E_2 is concentrated on the $s = 0$ line and so there is no room for differentials. Hence $E_2 \cong E_{\infty} \cong \text{Cotor}_{\Sigma}^{*,*}(S, N)$. \square

Proposition 5.14. (1) *Let $F \xrightarrow{\phi} G$ be a strict isomorphism of formal A -modules. Let F have a formal B -module structure compatible with its underlying formal A -module structure. Then, with the formal B -structure on G induced by ϕ , ϕ is a strict isomorphism of formal B -modules.*

In other words, the map of Hopf algebroids $(L^A, L^{AT}) \rightarrow (L^B, L^{BT})$ factors through $(L^B, L^B \otimes_{L^A} L^{AT})$.

(2) *Let L/K be totally ramified and let $F \xrightarrow{\phi} G$ be a strict isomorphism of A -typical formal A -modules. Let F have a B -typical formal B -module structure compatible with its underlying formal A -module structure. Then, with the formal B -structure on G induced by ϕ , ϕ is a strict isomorphism of B -typical formal B -modules.*

In other words, when L/K is totally ramified, the map of Hopf algebroids $(V^A, V^AT) \rightarrow (V^B, V^BT)$ factors through $(V^B, V^B \otimes_{V^A} V^AT)$.

Proof. In both cases, the map ϕ is a strict isomorphism of formal B -modules if $\phi(\rho_F(\alpha))(X) = \rho_G(\alpha)(\phi(X))$. But due to the way that ρ_G is defined, we have $\rho_G(\alpha)(\phi(X)) = \phi(\rho_F(\alpha)(\phi^{-1}(\phi(X))) = \phi(\rho_F(\alpha))(X)$. \square

We recall that a morphism of formal A -module laws over a commutative A -algebra R is just a power series $R[[X]]$ satisfying appropriate axioms. With this in mind, it is not a priori impossible for the same power series to be a morphism $F_1 \rightarrow G_1$ and also a morphism $F_2 \rightarrow G_2$ of formal A -module laws with $F_1 \neq F_2$ or $G_1 \neq G_2$. We want to rule out at least a certain case of this:

Lemma 5.15. *Let R be a commutative A -algebra and let $\phi \in R[[X]]$ be a power series such that it is an isomorphism $F \xrightarrow{\phi} G_1$ and an isomorphism $F \xrightarrow{\phi} G_2$ of formal A -module laws. Then $G_1 = G_2$.*

Proof. The identity map $(\phi \circ \phi^{-1})(X) = X$ is a morphism $G_1 \rightarrow G_2$, so

$$(\phi \circ \phi^{-1})G_1(X, Y) = G_2((\phi \circ \phi^{-1})(X), (\phi \circ \phi^{-1})(Y)),$$

but $(\phi \circ \phi^{-1})(X) = X$, so $G_1(X, Y) = G_2(X, Y)$; similarly, $(\phi \circ \phi^{-1}) \circ (\rho_{G_1}(\alpha)) = (\rho_{G_2}(\alpha)) \circ (\phi \circ \phi^{-1})$, so $\rho_{G_1} = \rho_{G_2}$. \square

Corollary 5.16. *Let K, L be p -adic number fields with rings of integers A, B , let L/K be a finite extension and let $F \xrightarrow{\phi} G$ be a strict isomorphism of formal B -module laws; let \tilde{G} denote the formal A -module law underlying G and let $\tilde{\phi}$ denote the strict isomorphism of formal A -module laws underlying ϕ ; finally, let G' be the formal B -module law induced by $F \xrightarrow{\tilde{\phi}} \tilde{G}$, using Def. 5.9. Then $G = G'$.*

Proof. The map ϕ is strict isomorphism of formal B -module laws $F \rightarrow G$ and also a strict isomorphism of formal B -module laws $F \rightarrow G'$, by Prop. 5.14; so by Lemma 5.15, $G = G'$. \square

Proposition 5.17. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be an extension.*

- (1) *The map $L^B \otimes_{L^A} L^A B \rightarrow L^B B$ from Prop. 5.14 is an isomorphism of graded commutative B -algebras, and the left unit maps $L^B \xrightarrow{\eta_L} L^B B$, $L^B \xrightarrow{\eta_L} L^B \otimes_{L^A} L^A B$ commute with this isomorphism.*
- (2) *When L/K is totally ramified, the map $V^B \otimes_{V^A} V^A T \rightarrow V^B T$ from Prop. 5.14 is an isomorphism of graded commutative B -algebras, and the left unit maps $V^B \xrightarrow{\eta_L} V^B T$, $V^B \xrightarrow{\eta_L} V^B \otimes_{V^A} V^A T$ commute with this isomorphism.*

Proof. (1) We first name some diagrams.

$$\left(\begin{array}{ccc} L^A & \longrightarrow & L^B \\ \downarrow \eta_L & & \\ L^A T & & \\ \uparrow \eta_R & & \\ L^A & \longrightarrow & L^A \end{array} \right) \xrightarrow{\Xi_1} \left(\begin{array}{ccc} L^A \otimes_A B & \longrightarrow & L^B \\ \downarrow \eta_L & & \\ L^A T \otimes_A B & & \\ \uparrow \eta_R & & \\ L^A \otimes_A B & \longrightarrow & L^A \otimes_A B \end{array} \right)$$

$$\xrightarrow{\Xi_2} \left(\begin{array}{ccc} L^B & \longrightarrow & L^B \\ \downarrow \eta_L & & \\ L^B T & & \\ \uparrow \eta_R & & \\ L^B & \longrightarrow & L^B \end{array} \right)$$

We will refer to the above three diagrams as X_1 , X_2 , and X_3 , respectively, and there are maps $X_1 \xrightarrow{\Xi_1} X_2 \xrightarrow{\Xi_2} X_3$ as indicated, which we have constructed in the previous propositions. The map Ξ_1 is a morphism of diagrams in the category of \mathbb{Z} -graded commutative \mathcal{O}_K -algebras, while Ξ_2 is a morphism of diagrams in the category of \mathbb{Z} -graded commutative B -algebras.

From Def. 5.9 and Cor. 5.16, we know that to specify a strict isomorphism of formal B -module laws is the same thing as to specify a source formal B -module law, a target formal A -module law, and a strict isomorphism of formal A -module laws, i.e., morphisms from X_2 to commutative B -algebras are in bijection with morphisms from X_3 to commutative B -algebras. So by the Yoneda Lemma, Ξ_2 induces an isomorphism of \mathbb{Z} -graded commutative \mathcal{O}_K -algebras on passing to colimits, and Ξ_1 does too. Since these isomorphisms were obtained by taking the colimits of the morphisms of the above diagrams in which the maps η_L appear, the isomorphisms commute with the left unit maps η_L .

- (2) We repeat the same argument as above, with the following diagrams:

$$\begin{aligned} & \left(\begin{array}{ccc} V^A & \longrightarrow & V^B \\ \downarrow \eta_L & & \\ V^A T & & \\ \uparrow \eta_R & & \\ V^A & \longrightarrow & V^A \end{array} \right) \xrightarrow{\Xi_1} \left(\begin{array}{ccc} V^A \otimes_A B & \longrightarrow & V^B \\ \downarrow \eta_L & & \\ V^A T \otimes_A B & & \\ \uparrow \eta_R & & \\ V^A \otimes_A B & \longrightarrow & V^A \otimes_A B \end{array} \right) \\ & \xrightarrow{\Xi_2} \left(\begin{array}{ccc} V^B & \longrightarrow & V^B \\ \downarrow \eta_L & & \\ V^B T & & \\ \uparrow \eta_R & & \\ V^B & \longrightarrow & V^B \end{array} \right) \end{aligned}$$

□

Corollary 5.18. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be an extension.*

- (1) *The map of Hopf algebroids $(L^A, L^A B) \longrightarrow (L^B, L^B B)$ classifying the underlying formal A -module structures on $(L^B, L^B B)$ is split, i.e., $(L^B, L^B B) \cong$*

$(L^B, L^B \otimes_{L^A} L^A B)$ as Hopf algebroids over B , and

$$\mathrm{Cotor}_{L^B B}^{*,*}(L^B, L^B) \cong \mathrm{Cotor}_{L^A B}^{*,*}(L^A, L^B).$$

- (2) If L/K is totally ramified, then the map of Hopf algebroids $(V^A, V^A T) \longrightarrow (V^B, V^B T)$ classifying the underlying formal A -module structures on $(V^B, V^B T)$ is split, i.e., $(V^B, V^B T) \cong (V^B, V^B \otimes_{V^A} V^A T)$ as Hopf algebroids over B , and

$$\mathrm{Cotor}_{V^B T}^{*,*}(V^B, V^B) \cong \mathrm{Cotor}_{V^A T}^{*,*}(V^A, V^B).$$

Proof. The splittings follow immediately from Prop. 5.17. The isomorphisms in cohomology follow immediately from Prop. 5.13. \square

Proposition 5.19. *Let $F \xrightarrow{\alpha} G$ be an isomorphism of formal A -module laws with logarithms, and let F be A -typical. Then G is A -typical if and only if*

$$\alpha^{-1}(X) = \sum_{i \geq 0} {}^F t_i X^{q^i}$$

for some collection $\{t_i \in A\}$ with $t_0 = 1$.

Proof. Assume that α^{-1} is of the above form.

$$\begin{aligned} \log_G(X) &= \log_F(\alpha^{-1}(X)) \\ &= \sum_{i \geq 0} \log_F(t_i X^{q^i}) \\ &= \sum_{i \geq 0} \sum_{j \geq 0} m_j (t_i X^{q^i})^{q^j} \end{aligned}$$

for some $\{m_j \in A \otimes_A K\}$, since F is A -typical. Reindexing,

$$= \sum_{i \geq 0} \left(\sum_{j=0}^i m_j t_{i-j}^{q^j} \right) X^{q^i},$$

so G is A -typical.

Now assume that F, G are both A -typical with logarithms $\log_F(X) = \sum_{i \geq 0} f_1(\ell_i^A) X^{q^i}$ and $\log_G(X) = \sum_{i \geq 0} f_2(\ell_i^A) X^{q^i}$. Now if we let

$$t_i = f_2(\ell_i^A) - \sum_{j=0}^{i-1} f_1(\ell_{i-j}^A) t_j^{q^{i-j}},$$

then

$$\begin{aligned} \log_G(X) &= \sum_{i \geq 0} \left(\sum_{j=0}^i f_1(\ell_i^A) t_{i-j}^{q^j} \right) X^{q^i} \\ &= \sum_{i \geq 0} \sum_{j \geq 0} f_1(\ell_j^A) (t_i X^{q^i})^{q^j} \\ &= \sum_{i \geq 0} \log_F(t_i X^{q^i}) \\ &= \log_F(\alpha^{-1}(X)). \end{aligned}$$

Hence

$$\begin{aligned}\alpha^{-1}(X) &= \log_F^{-1} \left(\sum_{i \geq 0} \log_F(t_i X^{q^i}) \right) \\ &= \sum_{i \geq 0} {}^F t_i X^{q^i}.\end{aligned}$$

□

The elements t_i in the previous lemma are precisely the images in R of the elements t_i^A under the map $V^A T \rightarrow R$ classifying the isomorphism α .

Proposition 5.20. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be an extension. Let ϕ denote the ring map*

$$L^B \xrightarrow{\phi} V^B \otimes_{V^A} V^A T$$

from Def. 5.9. Then the square

$$\begin{array}{ccc} L^B & \xrightarrow{\quad} & V^B \\ \downarrow \phi & & \downarrow \\ V^B \otimes_{V^A} V^A T & \longrightarrow & V^B T \end{array}$$

is a pushout square in the category of commutative B -algebras; in other words,

$$V^B T \cong (V^B \otimes_{V^A} V^A T) \otimes_{L^B} V^B$$

as commutative B -algebras.

Proof. The ring $V^B T$ classifies strict isomorphisms of B -typical formal B -module laws. To specify a strict isomorphism f of B -typical formal B -module laws is the same as to specify a B -typical formal B -module law F , and a strict isomorphism α of A -typical formal A -module laws from the underlying formal A -module law of F to some other A -typical formal A -module law G , such that, when we transport the formal B -module law structure across α to G (using Def. 5.9), the resulting formal B -module law is B -typical. In other words, a strict isomorphism of B -typical formal B -module laws over a commutative B -algebra R is specified by a ring map $V^B \otimes_{V^A} V^A T \rightarrow R$ such that the composite ring map

$$L^B \xrightarrow{\phi} V^B \otimes_{V^A} V^A T \rightarrow R$$

factors through the ring map $L^B \rightarrow V^B$. Such ring maps are precisely the ring maps

$$V^B T \cong (V^B \otimes_{V^A} V^A T) \otimes_{L^B} V^B \rightarrow R.$$

□

Corollary 5.21. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be an unramified extension with residue degree f . Then the ring map*

$$V^B \otimes_{V^A} V^A T \rightarrow V^B T$$

is surjective, with kernel the ideal generated by the elements

$$\{t_i^A : f \nmid i\} \subseteq V_B \otimes_{V^A} V^A T.$$

In other words,

$$V^B T \cong (B \otimes_A V^A T) / (\{v_i, t_i : f \nmid i\}).$$

Proof. Suppose the residue field of K is \mathbb{F}_q . We identify $V^B T$ with $(V^B \otimes_{V^A} V^A T) \otimes_{L^B} V^B$ using Prop. 5.20. By Prop. 5.19, a ring map $V^B \otimes_{V^A} V^A T \xrightarrow{g} R$ classifies a strict isomorphism $F \xrightarrow{\alpha} G$ of A -typical formal A -module laws such that

$$\alpha^{-1}(X) = \sum_{i \geq 0}^F g(t_i^A) X^{q^i},$$

and g factors through $V^B \otimes_{V^A} V^A T \rightarrow (V^B \otimes_{V^A} V^A T) \otimes_{L^B} V^B$ if and only if

$$\alpha^{-1}(X) = \sum_{i \geq 0}^F g(t_{f^i}^A) X^{q^{f^i}},$$

i.e., if and only if $g(t_i^A) = 0$ when $f \nmid i$. Hence

$$\begin{aligned} (V^B \otimes_{V^A} V^A T) \otimes_{L^B} V^B &\cong (V^B \otimes_{V^A} V^A T) / (\{t_i^A : f \nmid i\}) \\ &\cong (B \otimes_A V^A T) / (\{v_i^A, t_i^A : f \nmid i\}). \end{aligned}$$

□

Corollary 5.22. *Let K, L be p -adic number fields with rings of integers A, B , and let L/K be an unramified extension of degree > 1 . Then the Hopf algebroid map*

$$(V^A, V^A T) \rightarrow (V^B, V^B T)$$

is not split, i.e., $V^B T$ is not isomorphic to $V^B \otimes_{V^A} V^A T$ as a $V^A T$ -comodule, but we nevertheless have an isomorphism of bigraded abelian groups

$$\begin{aligned} \text{Ext}_{\text{graded } V^A T\text{-comodules}}^*(\Sigma^* V^A, (V^A T \otimes_{V^A} V^B) \square_{V^B T} M) \\ \cong \text{Ext}_{\text{graded } V^B T\text{-comodules}}^*(\Sigma^* V^B, M) \end{aligned}$$

for any graded $V^B T$ -comodule M which is flat over V^B .

Proof. It is a theorem of Ravenel (A1.3.12 in [7]) that, if $(R, \Gamma) \rightarrow (S, \Sigma)$ is a map of graded connected Hopf algebroids such that $\Gamma \otimes_R S \rightarrow \Sigma$ is surjective, and the inclusion map

$$(\Gamma \otimes_R S) \square_{\Sigma} B \subseteq \Gamma \otimes_R S$$

admits a retraction in the category of B -modules, then the change-of-rings spectral sequence

$$\text{Cotor}_{\Gamma}^{*,*}(R, \text{Cotor}_{\Sigma}^{*,*}(\Gamma \otimes_R S, M)) \Rightarrow \text{Cotor}_{\Sigma}^{*,*}(S, M)$$

collapses on to the 0-line at E_2 , giving an isomorphism of bigraded abelian groups

$$\text{Cotor}_{\Gamma}^{*,*}(R, (\Gamma \otimes_R S) \square_{\Sigma} M) \cong \text{Cotor}_{\Sigma}^{*,*}(S, M)$$

for any graded Σ -comodule M which is flat over S .

In the case of the Hopf algebroid map $(V^A, V^A T) \rightarrow (V^B, V^B T)$, Cor. 5.20 gives us that $V^B \otimes_{V^A} V^A T \rightarrow V^B T$ is surjective, and Cor. 4.2 gives us that $V^B \otimes_{V^A} V^A T$ is free as a V^B -module, so

$$(V^B \otimes_{V^A} V^A T) \square_{V^B T} V^B \subseteq V^B \otimes_{V^A} V^A T$$

admits a retraction in the category of V^B -modules. □

6. AUTOMORPHISM GROUP SCHEMES OF GEOMETRIC POINTS, UNDER CHANGE OF A .

In the earlier paper [11] we constructed an A -height stratification of the moduli stack $\mathcal{M}_{f mA}$ of one-dimensional formal A -modules, proved that each stratum has exactly one geometric point not in the next smaller stratum, and computed a presentation for the commutative Hopf algebra co-representing the automorphism group scheme of each positive, finite A -height geometric point of $\mathcal{M}_{f mA}$. These Hopf algebras are the Morava stabilizer algebras

$$\mathbb{F}_q \otimes_{V^A} V^A T \otimes_{V^A} \mathbb{F}_q$$

for formal A -module laws, which, when $A = \hat{\mathbb{Z}}_p$, are the starting point for computations of stable homotopy groups of spheres, using the chromatic computational machinery; see chapter 6 of [7].

However, the presentation computed in [11] is not functorial in A . In this section we compute the map between Hopf algebras co-representing automorphism group schemes of positive finite height induced by an extension of p -adic number fields. The main results are Prop. 6.6, which describes the map of Morava stabilizer algebras induced by a totally and tamely ramified field extension, and Cor. 6.8, which describes the map of Morava stabilizer algebras induced by an unramified field extension. For example, when K is an unramified extension of \mathbb{Q}_p of degree f and with ring of integers A , then the height n Morava stabilizer algebra is isomorphic to

$$(6.9) \quad \mathbb{F}_{p^n}[t_1, t_2, t_3, \dots] / (t_i^{p^n} - t_i : i \geq 1)$$

as a ring, while the classifying Hopf algebra of automorphisms of an A -height n/f formal A -module law over \mathbb{F}_{p^n} is the quotient Hopf algebra of 6.9:

$$\mathbb{F}_{p^n}[t_f, t_{2f}, t_{3f}, \dots] / (t_{if}^{p^n} - t_{if} : i \geq 1).$$

On the other hand, if K/\mathbb{Q}_p is totally and tamely ramified of degree $e > 1$ with uniformizer π , then let $\alpha = \frac{p}{\pi^e}$ and let $\epsilon(\alpha)$ be its image in the residue field \mathbb{F}_p . Let c be a primitive $\frac{p^n-1}{p^{n/e}-1}$ th root of $\epsilon(\alpha)$ in \mathbb{F}_{p^n} . Then the classifying Hopf algebra of automorphisms of an A -height n/e formal A -module law over \mathbb{F}_{p^n} is the quotient Hopf algebra of 6.9

$$\mathbb{F}_{p^n}[t_1, t_2, t_3, \dots] / (t_i^{p^{n/e}} - c^{q^i-1} t_i : i \geq 1).$$

This gives a number-theoretic description for a large class of the quotient Hopf algebras of the Morava stabilizer algebras described in Thm. 6.2.13 of [7].

If L/K is an extension of p -adic number fields of degree d and h is an integer multiple of d , then L embeds into the division algebra $D_{1/h, K}$ (see e.g. Serre's treatment of local class field theory in [1]), and furthermore, the embedding of L into $D_{1/h, K}$ extends to every field extension of L of degree h/d ; this gives us an injective ring morphism $D_{d/h, L} \rightarrow D_{1/h, K}$. The groups of units in the maximal orders of these division algebras are profinite groups. The following results are from [11]:

Proposition 6.1. *Let K be a p -adic number field with ring of integers A , and let e, f be the ramification degree and residue degree, respectively, of K/\mathbb{Q}_p . Let h be a positive integer and let F be a one-dimensional formal A -module law over $\text{Spec } \mathbb{F}_{p^f}$ of A -height h .*

- (1) The (not-necessarily-strict) automorphism group scheme of F is an $\mathbb{F}_{p^{efh}}/\mathbb{F}_{p^h}$ -twist of the constant profinite group scheme $\mathcal{O}_{D_{1/h,K}}^\times$.
- (2) After base change to $\mathbb{F}_{p^{efh}}$, we have a commutative diagram of constant profinite group schemes (or, what comes to the same thing, profinite groups) with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \text{strictAut}(F) & \longrightarrow & \text{Aut}(F) & \longrightarrow & \mathbb{Z}/(p^{fh} - 1)\mathbb{Z} & \longrightarrow & 1 \\
\downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
1 & \longrightarrow & \text{Syl}_p(\mathcal{O}_{D_{1/h,K}}^\times) & \longrightarrow & \mathcal{O}_{D_{1/h,K}}^\times & \longrightarrow & \mathbb{F}_{p^{fh}}^\times & \longrightarrow & 1
\end{array}$$

where by Syl_p of a profinite group, we mean the pro- p -Sylow subgroup, i.e., the pro- p -group obtained by taking the limit of the p -Sylow subgroup of each of the finite quotients of the original profinite group.

- (3) After base change to $\mathbb{F}_{p^{efh}}$, the Hopf algebra co-representing the strict automorphism group scheme of F is isomorphic to

$$\mathbb{F}_{p^{efh}}[t_1^A, t_2^A, \dots] / \left((t_i^A)^{p^{fh}} - t_i^A \text{ for each } i > 0 \right)$$

with coproduct given by

$$\Delta(t_k^A) = \sum_{i=0}^k t_i^A \otimes (t_{k-i}^A)^{q^i} + \sum_{j=e}^{\lfloor k/h \rfloor} w_j^A (\Delta_{k-ih}^A)^{q^{i(h-1)}},$$

where $\lfloor k/h \rfloor$ is the greatest integer less than or equal to k/h , and the sets Δ_{k-ih}^A are defined in [11]. We will write $S^A(h)$ for this Hopf algebra over $\mathbb{F}_{p^{efh}}$.

We know that

$$S^A(h) \cong \mathbb{F}_{p^{efh}}[t_1^A, t_2^A, \dots] / \left((t_i^A)^{p^{fh}} - t_i^A \text{ for each } i > 0 \right)$$

but for the purposes of computing the cohomology of the profinite groups $\mathcal{O}_{D_{1/h,K}}^\times$ (which relies on being able to compute the image of certain cohomology classes under the restriction maps induced by the inclusion $\mathcal{O}_{D_{d/h,L}}^\times \hookrightarrow \mathcal{O}_{D_{1/h,K}}^\times$, when $[L : K] = d$) we need to know how t_i^A relates to t_i^B when B is the ring of integers in an extension field of the fraction field K of A . In other words, we want to express $S^B(h/d)$ as a quotient of $S^A(h)$. We will be able to accomplish this for all field extensions L/K which are not wildly ramified; the totally and tamely ramified case is Prop. 6.6, and the unramified case is Cor. 6.8. First, we need to introduce equivariant thickenings of geometric points:

Definition 6.2. Let A be the completion at p of a number ring, let h be a positive integer, and let F be the one-dimensional formal A -module law over the residue field \mathbb{F}_q of A given by the ring-morphism

$$\begin{aligned}
V^A &\longrightarrow \mathbb{F}_q \\
V_i^A &\mapsto \begin{cases} 0 & \text{if } i \neq h \\ 1 & \text{if } i = h. \end{cases}
\end{aligned}$$

This classifying map factors through the \mathbb{Z} -graded ring-morphism

$$\begin{aligned} V^A &\longrightarrow \mathbb{F}_q[(V_h^A)^{\pm 1}] \\ V_i^A &\mapsto \begin{cases} 0 & \text{if } i \neq h \\ V_h^A & \text{if } i = h. \end{cases} \end{aligned}$$

With this V^A -algebra structure on the ring $\mathbb{F}_q[(V_h^A)^{\pm 1}]$, we will refer to $\mathbb{F}_q[(V_h^A)^{\pm 1}]$ as a \mathbb{G}_m -equivariant thickening of F .

Let L/K be a finite extension of p -adic number fields and let A, B be the rings of integers and $\mathbb{F}_q, \mathbb{F}_r$ the residue fields of K, L , respectively, and let h, j be positive integers. Then we define a map $\mathbb{F}_q[(v_h^A)^{\pm 1}] \rightarrow \mathbb{F}_r[(v_j^B)^{\pm 1}]$ by sending v_h^A to the image of v_h^A under the map $V^A \xrightarrow{\gamma} V^B \rightarrow \kappa^B(j)$.

See [11] for some discussion of the notion of “equivariant thickening” (in particular, the reason why this is an appropriate name for such an object), as well as the proof of the following proposition:

Proposition 6.3. *Let A be the completion of a number ring at a prime p , and let \mathbb{F}_q be the residue field of A . Let h be a positive integer. Then we have the isomorphism*

$$\begin{aligned} &\mathbb{F}_q[(V_h^A)^{\pm 1}] \otimes_{V^A} V^A T \otimes_{V^A} \mathbb{F}_q[(V_h^A)^{\pm 1}] \\ &\cong \mathbb{F}_q[(V_h^A)^{\pm 1}][t_1^A, t_2^A, \dots] / (\{t_i^A(v_h^A)^{q^i} - v_h^A(t_i^A)^{q^h} : i \text{ is a positive integer}\}). \end{aligned}$$

Proof. Tensoring $V^A T$ on the left with $\mathbb{F}_q[(V_h^A)^{\pm 1}]$ produces $\mathbb{F}_q[(V_h^A)^{\pm 1}] \otimes_{V^A} V^A T \cong \mathbb{F}_q[(V_h^A)^{\pm 1}][t_1^A, t_2^A, \dots]$. The right unit formula in [11] gives us that

$$V^A \xrightarrow{\eta_R} \mathbb{F}_q[(V_h^A)^{\pm 1}] \otimes_{V^A} V^A T$$

is determined by

$$\sum_{i \geq 0} {}^F t_i^A \eta_R(v_h^A)^{q^i} = \sum_{j \geq 0} {}^F v_h^A (t_j^A)^{q^h},$$

and in each of these formal sums there is only one element in each grading. Matching gradings, we get $t_i^A \eta_R(v_h^A)^{q^i} = v_h^A (t_i^A)^{q^h}$ in $\mathbb{F}_q[(V_h^A)^{\pm 1}] \otimes_{V^A} V^A T \otimes_{V^A} \mathbb{F}_q[(V_h^A)^{\pm 1}]$, giving us the relation in the statement of the theorem. \square

If L/K is a totally ramified extension of p -adic number fields with residue field \mathbb{F}_q and A and B are the rings of integers of K and L , then we have the composite ring map

$$V^A \rightarrow V^B \rightarrow \mathbb{F}_q[(v_j^B)^{\pm 1}]$$

for any positive integer j . For any positive integer h , we can take the image of v_h^A under the above composite ring map, and by sending v_h^A to that same image, get a ring map

$$\mathbb{F}_q[(v_h^A)^{\pm 1}] \xrightarrow{\kappa} \mathbb{F}_q[(v_j^B)^{\pm 1}].$$

This ring map may or may not be zero, depending on the extension L/K and the choice of integers h, j . In the next proposition we prove the only fact we will use about this ring map.

Proposition 6.4. *Let K, L be p -adic number fields, let L/K be a totally ramified extension of degree $n > 1$, and fix a positive integer j . Let A, B be the rings of*

integers of K, L , respectively, and let \mathbb{F}_q be their residue field. The least h such that the map $\mathbb{F}_q[(v_h^A)^{\pm 1}] \xrightarrow{\kappa} \mathbb{F}_q[(v_j^B)^{\pm 1}]$ is nonzero is $h = jn$. For $h = jn$ we have

$$(6.10) \quad \kappa(v_{jn}^A) = \epsilon \left(\frac{\pi_A - \pi_A^{q^{jn}}}{\prod_{m=1}^n (\pi_B - \pi_B^{q^{mj}})} \right) (v_j^B)^{\frac{q^{jn}-1}{q^j-1}}$$

$$(6.11) \quad = \epsilon \left(\frac{\pi_A}{\pi_B^n} \right) (v_j^B)^{\frac{q^{jn}-1}{q^j-1}},$$

where ϵ is reduction modulo the maximal ideal in B .

Proof. Assume $\mathbb{F}_q[(v_{h'}^A)^{\pm 1}] \rightarrow \mathbb{F}_q[(v_j^B)^{\pm 1}]$ is zero for all $h' < h$. Using Prop. 2.4 we have

$$\kappa(v_h^A) = \frac{\pi_A - \pi_A^{q^h}}{\pi_B - \pi_B^{q^h}} v_h^B + \sum_{i_1 + \dots + i_r = h \text{ and } r > 1} \left(\frac{\pi_A - \pi_A^{q^h}}{\pi_B - \pi_B^{q^h}} v_{i_1}^B \prod_{m=2}^r \frac{(v_{i_m}^B)^{q^{\sum_{s=1}^{m-1} i_s}}}{\pi_B - \pi_B^{q^{\sum_{s=1}^{m-1} i_s}}} - \kappa(v_{i_1}^A) \prod_{m=2}^r \frac{\kappa(v_{i_m}^A)^{q^{\sum_{s=1}^{m-1} i_s}}}{\pi_A - \pi_A^{q^{\sum_{s=1}^{m-1} i_s}}} \right),$$

which is zero in $\mathbb{F}_q[(v_j^B)^{\pm 1}]$ unless $j \mid h$. Assume $j \mid h$; then we have

$$\begin{aligned} \kappa(v_h^A) &= \frac{\pi_A - \pi_A^{q^h}}{\pi_B - \pi_B^{q^h}} v_j^B \prod_{m=2}^{h/j} \frac{(v_j^B)^{q^{(m-1)j}}}{\pi_B - \pi_B^{q^{(m-1)j}}} \\ &= \frac{\pi_A - \pi_A^{q^h}}{\prod_{m=1}^{h/j} (\pi_B - \pi_B^{q^{mj}})} (v_j^B)^{\frac{q^h-1}{q^j-1}} \end{aligned}$$

and $\left| \frac{\pi_A - \pi_A^{q^h}}{\prod_{m=1}^{h/j} (\pi_B - \pi_B^{q^{mj}})} \right|_p \geq 1$ if and only if $\frac{h}{j} \geq n$; since, in this case, $r = \frac{h}{j} > 1$

and $n > 1$, we cannot have $\frac{h}{j} > n$, so the lowest h such that we get a nonzero map $\mathbb{F}_q[(v_h^A)^{\pm 1}] \rightarrow \mathbb{F}_q[(v_j^B)^{\pm 1}]$ is $h = jn$. This gives us equation 6.10. A similar argument using the formula in Prop. 2.4 in terms of Hazewinkel generators instead of Araki generators gives us equation 6.11. \square

Corollary 6.5. *Let L/K be totally ramified of degree $e > 1$ and fix a positive integer h divisible by e . Let A, B be the rings of integers and of K, L , respectively, and let \mathbb{F}_q be their residue field. Then, from the maps $\mathbb{F}_q[(v_h^A)^{\pm 1}] \rightarrow \mathbb{F}_q[(v_{h/e}^B)^{\pm 1}]$, $V^A \rightarrow V^B$, and $V^A T \rightarrow V^B T$, we have a commutative diagram of commutative*

rings:

$$\begin{array}{ccccc}
 & & \mathbb{F}_q[(v_h^A)^{\pm 1}] & \longrightarrow & \mathbb{F}_q[(v_{h/e}^B)^{\pm 1}] \\
 & \nearrow & & \nearrow & \\
 V^A & \longrightarrow & V^B & & \\
 & \searrow & & \searrow & \\
 & & V^{AT} & \longrightarrow & V^{BT} \\
 & \nearrow & & \nearrow & \\
 V^B & \longrightarrow & V^B & & \\
 & \searrow & & \searrow & \\
 & & \mathbb{F}_q[(v_h^A)^{\pm 1}] & \longrightarrow & \mathbb{F}_q[(v_{h/e}^B)^{\pm 1}]
 \end{array}$$

and on taking colimits, this gives us the commutative diagram of Hopf algebras

$$\begin{array}{ccc}
 \mathbb{F}_q[(v_h^A)^{\pm 1}] \otimes_{V^A} V^{AT} \otimes_{V^A} \mathbb{F}_q[(v_h^A)^{\pm 1}] & & \\
 \downarrow & & \nearrow \\
 \mathbb{F}_q[(v_{h/e}^B)^{\pm 1}] \otimes_{V^B} V^{BT} \otimes_{V^B} \mathbb{F}_q[(v_{h/e}^B)^{\pm 1}] & \cong & \\
 \nearrow & & \downarrow \\
 \mathbb{F}_q[(v_h^A)^{\pm 1}][t_1^A, t_2^A, \dots] / (t_i^A (v_h^A)^{q^i} - v_h^A (t_i^A)^{q^h}) & & \\
 \downarrow g & & \\
 \mathbb{F}_q[(v_{h/e}^B)^{\pm 1}][t_1^B, t_2^B, \dots] / (t_i^B (v_{h/e}^B)^{q^i} - v_{h/e}^B (t_i^B)^{q^{h/e}}) & &
 \end{array}$$

in which

$$g(v_h^A) = \epsilon \left(\frac{\pi_K}{\pi_L^e} \right) \left(v_{h/e}^B \right)^{\frac{q^h - 1}{q^{h/e} - 1}},$$

where ϵ is $A \xrightarrow{\epsilon} \mathbb{F}_q$, reduction modulo the maximal ideal.

Proposition 6.6. *Let L/K be totally and tamely ramified of degree $e > 1$ and fix a positive integer h divisible by e . Let A, B be the rings of integers and of K, L , respectively, and let \mathbb{F}_q be their residue field. Let $\alpha = \frac{\pi_K}{\pi_L^e}$ and let c be a primitive $\frac{q^h - 1}{q^{h/e} - 1}$ th root of $\epsilon(\alpha)$ in \mathbb{F}_{q^h} . Then, we tensor the map g above on both sides over $\mathbb{F}_q[(v_h^A)^{\pm 1}]$ with \mathbb{F}_q , using the $\mathbb{F}_q[(v_h^A)^{\pm 1}]$ -algebra structure on \mathbb{F}_q from Def. 6.2, to*

get the commutative diagram of Hopf algebras

$$\begin{array}{ccc}
 \mathbb{F}_{q^h} \otimes_{V^A} V^A T \otimes_{V^A} \mathbb{F}_{q^h} & & \\
 \downarrow & & \searrow \\
 \mathbb{F}_{q^h} \otimes_{V^B} V^B T \otimes_{V^B} \mathbb{F}_{q^h} & & \cong \\
 \swarrow & & \downarrow \\
 \mathbb{F}_{q^h}[t_1^A, \dots] / ((t_i^A)^{q^h} - t_i^A) & & \\
 \downarrow \theta & & \swarrow \\
 \mathbb{F}_{q^h}[t_1^A, \dots] / ((t_i^A)^{q^{h/e}} - c^{q^i-1} t_i^A) & & \cong \\
 \swarrow & & \downarrow \\
 S^A(h) & & \\
 \downarrow & & \swarrow \\
 S^B(h/e) \otimes_{\mathbb{F}_{q^{h/e}}} \mathbb{F}_{q^h} & &
 \end{array}$$

where the map θ is the map g , from Cor. 6.5, tensored on both sides over $\mathbb{F}_q[(v_h^A)^{\pm 1}]$ with \mathbb{F}_q , using the $\mathbb{F}_q[(v_h^A)^{\pm 1}]$ -algebra structure on \mathbb{F}_q from Def. 6.2. We have $\theta(t_i^A) = t_i^A$.

Proof. From Cor. 6.5 we know that $g(v_h^A) = \frac{1}{\epsilon(\alpha)} \left(v_{h/e}^B \right)^{\frac{q^h-1}{q^{h/e}-1}}$, so imposing the relation $v_h^A = 1$ on $S^A(h)$ also imposes the relation $v_{h/e}^B = c$ on $S^B(h/e) \otimes_{\mathbb{F}_{q^{h/e}}} \mathbb{F}_{q^h}$, for some primitive $\frac{q^h-1}{q^{h/e}-1}$ th root of $\epsilon(\alpha)$ in \mathbb{F}_{q^h} ; so the relation $t_i^B (v_{h/e}^B)^{q^i} = v_{h/e}^B (t_i^B)^{q^{h/e}}$ in $\mathbb{F}_q[(v_{h/e}^B)^{\pm 1}] \otimes_{V^B} V^B T \otimes_{V^B} \mathbb{F}_q[(v_{h/e}^B)^{\pm 1}]$ becomes the relation $(t_i^B)^{q^{h/e}} = c^{q^i-1} t_i^B$ in $S^B(h/e) \otimes_{\mathbb{F}_{q^{h/e}}} \mathbb{F}_{q^h}$. Now our map θ classifies the strict automorphism of formal A -module laws underlying any strict automorphism of the formal B -module law classified by the map $V^B \rightarrow \mathbb{F}_{q^h}$ sending $v_{h/e}^B$ to 1 and all other generators v_i^B to zero; so, by Prop. 5.19, θ sends t_i^A to t_i^B . This gives us the description of $S^B(h/e) \otimes_{\mathbb{F}_{q^{h/e}}} \mathbb{F}_{q^h}$ as a quotient Hopf algebra of $S^A(h)$ given above. \square

Proposition 6.7. *Let L/K be unramified of degree $f > 1$ and fix a positive integer h divisible by f . Let A, B be the rings of integers and of K, L , respectively, and let \mathbb{F}_q be the residue field of K . Then, from the maps $\mathbb{F}_q[(v_h^A)^{\pm 1}] \rightarrow \mathbb{F}_{q^f}[(v_{h/f}^B)^{\pm 1}]$,*

$V^A \rightarrow V^B$, and $V^A T \rightarrow V^B T$, we have a commutative diagram of Hopf algebras:

$$\begin{array}{ccccc}
 & & \mathbb{F}_q[(v_h^A)^{\pm 1}] & \longrightarrow & \mathbb{F}_{q^f}[(v_{h/f}^B)^{\pm 1}] \\
 & \nearrow & & \nearrow & \\
 V^A & \longrightarrow & V^B & & \\
 & \searrow & & \searrow & \\
 & & V^A T & \longrightarrow & V^B T \\
 & \nearrow & & \nearrow & \\
 V^B & \longrightarrow & V^B & & \\
 & \searrow & & \searrow & \\
 & & \mathbb{F}_q[(v_h^A)^{\pm 1}] & \longrightarrow & \mathbb{F}_{q^f}[(v_{h/f}^B)^{\pm 1}]
 \end{array}$$

and on taking colimits, this gives us the commutative diagram of Hopf algebras

$$\begin{array}{ccc}
 \mathbb{F}_q[(v_h^A)^{\pm 1}] \otimes_{V^A} V^A T \otimes_{V^A} \mathbb{F}_q[(v_h^A)^{\pm 1}] & & \\
 \downarrow & \searrow & \\
 \mathbb{F}_{q^f}[(v_{h/f}^B)^{\pm 1}] \otimes_{V^B} V^B T \otimes_{V^B} \mathbb{F}_{q^f}[(v_{h/f}^B)^{\pm 1}] & \xrightarrow{\cong} & \\
 \cong \swarrow & & \searrow \\
 \mathbb{F}_q[(v_h^A)^{\pm 1}][t_1^A, t_2^A, \dots] / (t_i^A (v_h^A)^{q^i} - v_h^A (t_i^A)^{q^h}) & & \\
 \downarrow g & & \\
 \mathbb{F}_{q^f}[(v_{h/f}^B)^{\pm 1}][t_1^B, t_2^B, \dots] / (t_i^B (v_{h/f}^B)^{q^{fi}} - v_{h/f}^B (t_i^B)^{q^h}) & &
 \end{array}$$

in which

$$g(v_h^A) = v_{h/e}^B$$

and

$$g(t_i^A) = \begin{cases} t_{i/f}^B & \text{if } f \mid i \\ 0 & \text{if } f \nmid i \end{cases}.$$

Proof. This is a consequence of Prop. 4.1, Prop. 5.19, and Prop. 6.3. \square

Corollary 6.8. *Let L/K be unramified and of degree $f > 1$ and fix a positive integer h divisible by f . Let A, B be the rings of integers and of K, L , respectively, and let \mathbb{F}_q be the residue field of A . Then, we tensor the map g above on both sides over $\mathbb{F}_q[(v_h^A)^{\pm 1}]$ with \mathbb{F}_q , using the $\mathbb{F}_q[(v_h^A)^{\pm 1}]$ -algebra structure on \mathbb{F}_q from*

Def. 6.2, to get the commutative diagram of Hopf algebras

$$\begin{array}{ccc}
 \mathbb{F}_{q^h} \otimes_{V^A} V^A T \otimes_{V^A} \mathbb{F}_{q^h} & & \\
 \downarrow & \searrow & \\
 \mathbb{F}_{q^h} \otimes_{V^B} V^B T \otimes_{V^B} \mathbb{F}_{q^h} & & \cong \\
 \swarrow & \downarrow & \swarrow \\
 \mathbb{F}_{q^h} [t_1^A, t_2^A, t_3^A, \dots] / ((t_i^A)^{q^h} - t_i^A) & & \\
 \downarrow \theta & & \\
 \mathbb{F}_{q^h} [t_f^A, t_{2f}^A, t_{3f}^A, \dots] / ((t_i^A)^{q^h} - t_i^A) & & \cong \\
 \swarrow & \downarrow & \swarrow \\
 S^A(h) & & \\
 \downarrow & & \\
 S^B(h/f) \otimes_{\mathbb{F}_{q^h/f}} \mathbb{F}_{q^h} & &
 \end{array}$$

where the map θ is the map g , from Cor. 6.5, tensored on both sides over $\mathbb{F}_q[(v_h^A)^{\pm 1}]$ with \mathbb{F}_q , using the $\mathbb{F}_q[(v_h^A)^{\pm 1}]$ -algebra structure on \mathbb{F}_q from Def. 6.2. We have

$$\theta(t_i^A) = \begin{cases} t_i^A & \text{if } f \mid i \\ 0 & \text{if } f \nmid i \end{cases}.$$

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